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Generalized Transforms and their Asymptotic Behaviour

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GENERALIZED TRANSFORMS AND THEIR ASYMPTOTIC BEHAVIOUR

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CONTENTS

Certain properties of generalized transforms of the type

$$\int_0^\infty g(x)\,h(\alpha, x)\,\mathrm{d}x$$

are derived when g is a generalized function in the terminology of Lighthill (1958) and Jones (1966*b*). The kernel function h is assumed to be smooth and of sufficiently slow growth at infinity for the generalized transform to exist for any generalized function g. Nevertheless, the class of kernel functions is wide and includes functions such as $\mathrm{e}^{\mathrm{i}\alpha x^2}$ and the Bessel function $J_n(\alpha x)$. Theorems concerning the derivative and the limit (in the generalized sense) of the generalized transform are established. The problem of the inversion of generalized transforms is also discussed.

The analogue of the Riemann–Lebesgue lemma for generalized transforms is obtained when g is a conventional function and the restrictions on h are relaxed so that it need only be the derivative of a function with suitable properties.

The asymptotic behaviour as $\alpha \to +\infty$ of the generalized transform is examined under the condition that g is infinitely differentiable (in the ordinary sense) at all but a finite number of points. It is shown that the main contribution to the asymptotic development comes from intervals near these points and the point at infinity. Criteria are provided which demonstrate that in many important practical cases the contribution from the point at infinity is essentially exponentially small and therefore negligible. The contributions from the other critical points are determined under a variety of circumstances. In all cases the aim has been to consider conditions which are likely to be of practical value, to be capable of relatively straightforward verification and yet yield theorems of reasonable utility and wide applicability.

Some illustrations of the applications of the theorems are given; they include Bessel functions, Laplace transforms and the Hankel transform.

1. Introduction

In an earlier paper (Jones 1966*a*) the author considered the asymptotic behaviour of Fourier integrals and the method of stationary phase, deriving some general theorems of wide applicability. The aim of the present paper is to provide corresponding theorems for integrands of a more extended class. Although the permitted class investigated is considerably more extensive than in the earlier paper, it has been found possible to obtain very general theorems involving only relatively simple assumptions. However, no attempt has been made to establish the theorems under the weakest possible conditions. Only circumstances which are likely to occur frequently in practice have been taken into account.

The primary objective of the work is the derivation of asymptotic formulae for integrals of the type

$$\int_0^\infty g(x)\,h(\alpha, x)\,\mathrm{d}x$$

as $\alpha \to +\infty$. Here g is a generalized function but the kernel function h is subject to certain restrictions, which will be stated later, to ensure that the integral has a meaning as a generalized function of α. (The terminology is that of Lighthill (1958) and Jones (1966*b*).) Such integrals may be regarded as generalized transforms of g. Although the main concern is with asymptotic behaviour it has been found convenient to develop some useful theorems dealing with the transform of g defined by the integral above. One of the most valuable attributes of a transform is the existence of an inversion theorem in suitable circumstances. However, in order to prevent undue restriction on the integrand, there is no insistence that an inversion theorem should be available, although there is some discussion of the conditions under which inversion is likely to be valid in § 7 (see also § 17).

Section 2 is concerned with specifying the conditions which are to be imposed on h and the properties which result. In addition, two important subclasses are introduced because they cover kernels of frequent occurrence and can lead to more precise estimates of the asymptotic behaviour. It is shown in § 3 that, for the given class of h, the generalized transform of any generalized function can be defined and that this definition is consistent with earlier definitions of narrower application. Theorems about the derivative of a generalized transform and the effect of a linear change of variable are obtained in § 4. It is then possible to determine the structure of a generalized transform—this is carried out in § 5. The generalized limit is discussed in § 6 and then some inversion properties are derived in § 7.

In the derivation of the asymptotic behaviour of the generalized transform for large α it is found helpful to have available theorems which are analogous to the Riemann–Lebesgue lemma. These theorems are derived in § 8, some attention being paid to an important class of functions for which better estimates can be obtained of the asymptotic behaviour of the integral than in the Riemann–Lebesgue lemma. Section 9 is concerned with deriving simple tests of whether or not a kernel function complies with the conditions required by the theory of § 8.

The class of generalized functions under consideration in § 10 is limited to those likely to be of most practical value. These generalized functions have a finite number of 'singularities' or critical points. It is shown in § 10 that the main contribution to the asymptotic behaviour of the generalized transform comes from the critical points. The critical point at infinity is examined in detail in § 11 and criteria are given under which the contribution is negligible. The effects of a discontinuity in a derivative are obtained in § 12, whereas the contribution from the critical point at the origin is derived under more general conditions in § 13. It is demonstrated in § 14 how the theory for a critical point which is at neither infinity nor the origin can be subsumed under the preceding theory. The problem of approximating the kernel function is investigated in § 15 and some useful theorems derived.

Sections 16 and 17 give some illustrations of the theory that has been developed; § 16 deals with asymptotic behaviour and § 17 with inversion of the Hankel transform.

Some results are given in appendices A and B; reference to equations there is indicated by the addition of the letter A or B respectively.

2. The kernel functions

Integrals of the type

$$\int_0^\infty g(x)\, h(\alpha, x)\, \mathrm{d}x$$

are of frequent occurrence in applied mathematics, especially when transform methods are employed. Usually these integrals cannot be evaluated exactly and it is necessary to resort to asymptotic techniques. In order that these techniques shall be of reasonably wide applicability it is desirable to consider the integral in a generalized sense. If g is a generalized function, some limitation will have to be placed on h to ensure that the integral exists, even in a generalized sense. This section is concerned with laying down suitable restriction on h, while still permitting a wide variety of h, and developing properties that will be useful subsequently.

Throughout the paper it will be assumed that α and x are real; indeed, in most of the analysis they are non-negative.

Definition 1. *We write $h \in \mathscr{H}^+$ when, on $\alpha \geqslant 0$ and $x \geqslant 0$, h is infinitely differentiable (in the ordinary sense) with respect to α and x,*

$$\left|\frac{\partial^{r+s} h}{\partial x^r\, \partial \alpha^s}\right| < C_{rs}(1+\alpha^2)^{a_{rs}}\,(1+x^2)^{b_{rs}} \quad (\alpha \geqslant 0,\ x \geqslant 0) \tag{1}$$

for $r, s = 0, 1, 2, \ldots$ and there exist u_r, v_r such that

$$\frac{\partial^r u_r}{\partial x^r} = h, \quad \left|\frac{\partial^q u_r}{\partial \alpha^q}\right| < C_{qr}(1+\alpha^2)^{M_q+a_1 r}\,(1+x^2)^{M_{qr}}/\alpha^{P_{qr}} \quad (\alpha > 0,\ x \geqslant 0), \tag{2}$$

$$\frac{\partial^r v_r}{\partial \alpha^r} = h, \quad \left|\frac{\partial^q v_r}{\partial x^q}\right| < C_{qr}(1+\alpha^2)^{N_{qr}}(1+x^2)^{N_q+a_2 r}/x^{Q_{qr}} \quad (\alpha \geqslant 0,\ x > 0), \tag{3}$$

for $r = 1, 2, \ldots$ and $q = 0, 1, 2, \ldots$. The real numbers a_1, a_2 satisfy $0 \leqslant a_1, a_2 < \frac{1}{2}$, the real numbers M_q and N_q are independent of r and

$$P_{q,r+1} - P_{qr} \geqslant 1, \quad Q_{q,r+1} - Q_{qr} \geqslant 1. \tag{4}$$

There are clearly connexions between some of the exponents in (1) to (3), e.g. $M_{qr} = b_{0,q-r}$ $(q \geqslant r)$. If u_0 and v_0 are identified with h it follows that

$$M_0 = a_{00} = N_{00}, \quad M_{00} = b_{00} = N_0,$$

when P_{00} and Q_{00} are zero. It will be remarked that considerable freedom is attached to the choice of a_{rs}, b_{rs}, M_q, M_{qr}, P_{qr}, N_q, N_{qr} and Q_{qr}.

In saying that h is differentiable at $x = 0$, we require only that the right derivative exists there; a similar convention is understood at $\alpha = 0$.

The symbol C_{qr} is used generically here, and subsequently, to indicate some finite non-negative number independent of α and x; it does not necessarily represent the same value at each occurrence.

A simple example of a function in $\mathscr{H}^+$ is $e^{i\alpha x}$. In this case $u_r = e^{i\alpha x}/(i\alpha)^r$ and $v_r = e^{i\alpha x}/(ix)^r$, which obviously satisfy the conditions imposed in definition 1. Another simple example is $h(\alpha, x) = e^{i\alpha x^2}$; here

$$u_r = (-)^r \int_x^\infty \int_{t_{r-1}}^\infty \dots \int_{t_1}^\infty e^{i\alpha t^2}\, dt\, dt_1 \dots dt_{r-1}, \quad v_r = \frac{e^{i\alpha x^2}}{(ix^2)^r}.$$

The conditions on v_r are obviously met. As for u_r, induction, with the use of

$$u_r = -\int_x^\infty u_{r-1}\, dt,$$

shows that u_r behaves like a constant multiple of $e^{i\alpha x^2}/(\alpha x)^r$ for large x and is $O(1/\alpha^{\frac{1}{2}r})$ for small x. The required properties follow immediately.

One attribute of $\mathscr{H}^+$ that will be important subsequently is contained in

THEOREM 1. *If $\chi(x)$ is infinitely differentiable (in the ordinary sense) on $x \geqslant 0$, if*

$$|\chi^{(r)}(x)| < C_r(1+x^2)^{P_r} \quad (x \geqslant 0) \tag{5}$$

for $r = 0, 1, 2, \dots$, and if $h \in \mathscr{H}^+$ then $\chi(x)\, h(\alpha, x) \in \mathscr{H}^+$.

Proof. By Leibniz's theorem

$$\left|\frac{\partial^{r+s}(\chi h)}{\partial x^r\, \partial \alpha^s}\right| < C \sum_{p=0}^{r} \left|\chi^{(p)} \frac{\partial^{r+s-p} h}{\partial x^{r-p}\, \partial \alpha^s}\right| < C \sum_{p=0}^{r} (1+\alpha^2)^{a_{r-p,s}}(1+x^2)^{b_{r-p,s}+P_p}$$

from (1) and (5). Evidently χh satisfies an inequality of the form (1), though not necessarily with the same a_{rs} and b_{rs}. It remains to demonstrate the existence of functions satisfying (2) and (3).

Let U_r be defined for $r = 1, 2, \dots$, by

$$U_r = \chi u_r + \frac{(-)^r}{(r-1)!} \int_0^x u_r(\alpha, t) \frac{\partial^r}{\partial t^r}\{(x-t)^{r-1}\chi(t)\}\, dt \tag{6}$$

when $\alpha > 0$. Then

$$\begin{aligned} \frac{\partial U_r}{\partial x} &= (1-r)\,\chi' u_r + \chi \frac{\partial u_r}{\partial x} + \frac{(-)^r}{(r-2)!} \int_0^x u_r(\alpha, t) \frac{\partial^r}{\partial t^r}\{(x-t)^{r-2}\chi(t)\}\, dt \\ &= u_r(\alpha, 0)\, P_{r-2}(x) + \chi \frac{\partial u_r}{\partial x} + \frac{(-)^{r-1}}{(r-2)!} \int_0^x \frac{\partial u_r(\alpha, t)}{\partial t} \frac{\partial^{r-1}}{\partial t^{r-1}}\{(x-t)^{r-2}\chi(t)\}\, dt \end{aligned} \tag{7}$$

after an integration by parts. Here P_{r-2} is a polynomial of degree $r-2$ in x whose coefficients do not involve α.

Apart from the first term (7) is the same as (6) with r replaced by $r-1$ except that u_r is replaced by $\partial u_r/\partial x$. Therefore, there is no difficulty in seeing that

$$\frac{\partial^r U_r}{\partial x^r} = \chi(x) \frac{\partial^r u_r}{\partial x^r} = \chi(x)\, h(\alpha, x).$$

Also, from (6), (5) and (2),

$$\left|\frac{\partial^q U_r}{\partial \alpha^q}\right| < C \frac{(1+\alpha^2)^{M_q + a_1 r}}{\alpha^{P_{qr}}} \left\{(1+x^2)^{P_0 + M_{qr}} + \int_0^x (1+t^2)^{M_{qr}} \left|\frac{\partial^r}{\partial t^r}\{(x-t)^{r-1}\chi(t)\}\right| dt\right\}. \tag{8}$$

From (8) and (5) it is evident that an inequality of the type (2) is satisfied; the main difference is that M_{qr} is replaced by sup $(M_{qr}+P_0, M_{qr}+P_1+\frac{1}{2}, \dots, M_{qr}+P_r+\frac{1}{2}r)$.

Correspondingly, $V_r = \chi v_r$ is such that $\partial^r V_r/\partial\alpha^r = \chi h$ and

$$\left|\frac{\partial^q V_r}{\partial x^q}\right| < C \sum_{m=0}^{q} (1+\alpha^2)^{N_{mr}} \frac{(1+x^2)^{N_m+a_2 r+P_{q-m}}}{x^{Q_{mr}}}$$

on $x > 0$. This is the same as (3) but with N_{qr}, Q_{qr}, N_q replaced by

$$\sup_m N_{mr}, \quad \sup_m Q_{mr}, \quad \sup_m (N_m + P_{q-m} - \tfrac{1}{2}Q_{mr} + \tfrac{1}{2}\sup_n Q_{nr}),$$

respectively, with m, n running through $0, 1, \ldots, q$. Thus V_r has the same properties as v_r and the proof of the theorem is complete.

The interchange of α and x in the above proof leads to the conclusion that $\chi(\alpha)\,h(\alpha, x) \in \mathscr{H}^+$ but this fact will not be needed subsequently.

It is also possible to multiply h by a function of α and x without leaving $\mathscr{H}^+$. One particular case of interest is given by

Theorem 2. *If $h \in \mathscr{H}^+$ and if*

$$\left|\frac{\partial^{r+s}\lambda}{\partial x^r\,\partial\alpha^s}\right| < \mathrm{e}^{-c_{rs}\alpha x}(1+\alpha^2)^{a'_{rs}}(1+x^2)^{b'_{rs}} \quad (c_{rs} > 0)$$

for $r, s = 0, 1, 2, \ldots$, then $h\lambda \in \mathscr{H}^+$.

Proof. Obviously $h\lambda$ complies with (1). Define

$$U_r = \frac{-1}{(r-1)!}\int_x^\infty h(\alpha, t)\,\lambda(\alpha, t)\,(x-t)^{r-1}\,\mathrm{d}t.$$

Then $\partial^r U_r/\partial x^r = h\lambda$ and

$$|U_r| < C(1+\alpha^2)^{a_{00}+a'_{00}} \int_x^\infty (1+t^2)^{b_{00}+b'_{00}}(t-x)^{r-1}\,\mathrm{e}^{-c_{00}\alpha t}\,\mathrm{d}t$$

$$< C(1+\alpha^2)^{a_{00}+a'_{00}}\,\mathrm{e}^{-c_{00}\alpha x}\int_0^\infty \{1+(t-x)^2\}^{b_{00}+b'_{00}}\,t^{r-1}\,\mathrm{e}^{-c_{00}\alpha t}\,\mathrm{d}t$$

$$< C\frac{(1+\alpha^2)^{a_{00}+b_{00}+a'_{00}+b'_{00}}}{(c_{00}\alpha)^{r+2b_{00}+2b'_{00}}}\,(1+x^2)^{b_{00}+b'_{00}}\,\mathrm{e}^{-c_{00}\alpha x}.$$

Similar considerations apply to the derivatives of U_r and also to V_r, defined analogously. Hence the theorem is proved.

There are two subclasses of $\mathscr{H}^+$ which will prove to be valuable in later sections. The first of these is defined as follows

Definition 2. *Let j be infinitely differentiable (in the ordinary sense) with respect to α and x on $\alpha \geqslant 0$, $x \geqslant 0$ and satisfy*

$$\left|\frac{\partial^{r+s}j}{\partial x^r\,\partial\alpha^s}\right| < C_{rs}(1+\alpha^2)^{c_1 r+b_s}(1+x^2)^{c_2 s+a_r} \quad (\alpha \geqslant 0, \quad x \geqslant 0) \tag{9}$$

for $r, s = 0, 1, 2, \ldots$, where $0 \leqslant c_1, c_2 < \frac{1}{2}$ and the numbers a_r, b_s are independent of s, r respectively. Then, if h is the sum of a finite number of terms of the type $j(\alpha, x)\,\mathrm{e}^{-\mathrm{i}b\alpha x}$ where b is real and non-zero, we write $h \in \mathscr{H}_1^+$.

Some examples of functions which are in $\mathscr{H}_1^+$ are

$$\cos\alpha x, \quad (1+\alpha^2+x^2)^{-1}\,\mathrm{e}^{-\mathrm{i}\alpha x}, \quad (1+\alpha^3+x^2)^{\frac{1}{2}}\,\mathrm{e}^{2\mathrm{i}\alpha x}, \quad \cos\{(1+\alpha+x)^{\frac{1}{2}}+\alpha x\}.$$

We wish to show that $h \in \mathscr{H}_1^+$ implies that $h \in \mathscr{H}^+$. Clearly it will be sufficient to demonstrate this for a term of the type $j\,\mathrm{e}^{-\mathrm{i}b\alpha x}$; the result for the sum of a finite number of such terms is then immediate.

On account of (9), (1) is satisfied with

$$a_{rs} = \sup\{c_1 m + b_n + \tfrac{1}{2}(r-m)\}, \quad b_{rs} = \sup\{c_2 n + a_m + \tfrac{1}{2}(s-n)\}$$

as m runs through $0, 1, \ldots, r$ and n runs through $0, 1, \ldots, s$. It remains to determine u_r and v_r.

When $\alpha > 0$, let

$$u_r = (-)^r \frac{j(\alpha, x)}{(\mathrm{i}b\alpha)^r}\,\mathrm{e}^{-\mathrm{i}b\alpha x} + \frac{1}{(r-1)!}\int_0^x \frac{\mathrm{e}^{-\mathrm{i}b\alpha t}}{(\mathrm{i}b\alpha)^r}\frac{\partial^r}{\partial t^r}\{j(\alpha, t)\,(x-t)^{r-1}\}\,\mathrm{d}t. \tag{10}$$

Then it is easy to see, after an integration by parts, that

$$\partial u_r/\partial x = u_{r-1} + P_{r-2}(x), \tag{11}$$

where P_{r-2} is a polynomial of degree $r-2$ in x. Since $\partial u_1/\partial x = j(\alpha, x)\,\mathrm{e}^{-\mathrm{i}b\alpha x}$ it follows that

$$\partial^r u_r/\partial x^r = j(\alpha, x)\,\mathrm{e}^{-\mathrm{i}b\alpha x}.$$

Furthermore, it is plain from (9) and (10) that

$$\left|\frac{\partial^q u_r}{\partial \alpha^q}\right| < C_{qr}(1+\alpha^2)^{M'_q + c_1 r}(1+x^2)^{M'_{qr}}/\alpha^{q+r}, \tag{12}$$

where
$$M'_q = \tfrac{1}{2}q + \sup(b_0, b_1, \ldots, b_q),$$
$$M'_{qr} = \tfrac{1}{2}q + \sup(a_0, a_1 + \tfrac{1}{2}, \ldots, a_r + \tfrac{1}{2}r).$$

Consequently, an inequality of the type (2) is met.

By interchanging the roles of α and x in the analysis a v_r is obtained which satisfies (3). Therefore, we have demonstrated

THEOREM 3. *If $h \in \mathscr{H}_1^+$ then $h \in \mathscr{H}^+$.*

An immediate deduction from theorem 1 is that, $\chi h \in \mathscr{H}^+$ if $h \in \mathscr{H}_1^+$, but more is true. For, from (5) and (9),

$$\left|\frac{\partial^{r+s}}{\partial x^r\,\partial \alpha^s}(\chi j)\right| < C_{rs}(1+\alpha^2)^{c_1 r + b_s}(1+x^2)^{c_2 s + a'_r}, \tag{13}$$

where $a'_r = \sup(a_0 + P_r, a_1 + P_{r-1}, \ldots, a_r + P_0)$. Inequality (13) is of the same form as (9) and so we can assert

THEOREM 4. *If $h \in \mathscr{H}_1^+$ and χ satisfies the condition of theorem 1, then $\chi(x)\,h(\alpha, x) \in \mathscr{H}_1^+$.*

The second subclass of $\mathscr{H}^+$ that is of interest stems from

DEFINITION 3. *Let $k(x)$ be infinitely differentiable (in the ordinary sense) on $x \geqslant 0$ and satisfy*

$$|k^{(r)}(x)| < C_r(1+x^2)^{N-\delta r} \quad (x \geqslant 0), \tag{14}$$

where $N \geqslant 0$ and $\delta > 0$. Then, if h is the sum of a finite number of terms of the type $k(\alpha x)\,\mathrm{e}^{-\mathrm{i}b\alpha x}$ where b is real and non-zero, we write $h \in \mathscr{H}_2^+$.

Typical examples of functions in $\mathscr{H}_2^+$ are

$$\mathrm{e}^{\mathrm{i}\alpha x}, \quad \cos\alpha x, \quad J_0(\alpha x), \quad \sin\alpha x \ln(1+\alpha x), \quad J_\nu(\alpha x)/(\alpha x)^\nu,$$
$$\mathrm{e}^{\mathrm{i}\{\alpha x + (\alpha x + 1)^{\frac{1}{2}}\}}, \quad (1+\alpha^2 x^2)^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}b\alpha x}.$$

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The space also includes $e^{-\alpha x}$ since e^{-x+ibx} satisfies (14) and therefore contains functions which cannot be regarded as oscillatory. The space could be restricted to functions, that could be called oscillatory, by replacing (14) by

$$|k^{(r+1)}(x)| < (1+x^2)^{-\delta}\,|k^{(r)}(x)|. \tag{15}$$

Such oscillatory functions come within the scope of (14) and, indeed, all the examples in the first sentence of the paragraph can be described as oscillatory in this sense. However, $e^{-\alpha x}$ would be excluded by (15).

There are two conclusions which can be drawn at once from definition 3 which, for future reference, will be stated as lemmas without proof.

LEMMA 1. *If* $|k^{(r)}(x)| < C_r e^{-a_r x}$ $(a_r > 0,\ x \geqslant 0)$ *for* $r = 0, 1, \ldots$ *then* $k(ax)\,e^{-ib\alpha x} \in \mathscr{H}_2^+$.

LEMMA 2. *If* $h \in \mathscr{H}_2^+$ *then* $h^{(n)}(\alpha x) \in \mathscr{H}_2^+$ *(n a non-negative integer) with the same values of N and* δ.

Before proceeding to show that $h \in \mathscr{H}_2^+$ implies that $h \in \mathscr{H}^+$ we note that h need not lie in $\mathscr{H}_1^+$. For consider $h(\alpha x) = e^{i\alpha x}/(1+\alpha x)$; this satisfies definition 3 with $k(x) = (1+x)^{-1}$. But $\partial^r k(\alpha x)/\partial x^r$ does not comply with (9) for any c less than $\frac{1}{2}$, as can be seen by examining the behaviour on $x = 1/\alpha$. Therefore $k(\alpha x)$ cannot be identified with $j(\alpha, x)$.

On the other hand, there are functions such as $\alpha^2x^2\,e^{i\alpha x}$ which are in $\mathscr{H}_2^+$ and $\mathscr{H}_1^+$. Obviously there are functions in $\mathscr{H}_1^+$ which are not of the form $h(\alpha x)$. Hence $\mathscr{H}_1^+$ and $\mathscr{H}_2^+$ are subclasses of $\mathscr{H}^+$ which intersect but neither is contained in the other. However, $\mathscr{H}_1^+$ and $\mathscr{H}_2^+$ do not exhaust $\mathscr{H}^+$ because $e^{i\alpha x^2}$ is in $\mathscr{H}^+$ but not in either $\mathscr{H}_1^+$ or $\mathscr{H}_2^+$.

To show that $h \in \mathscr{H}^+$ follows from $h \in \mathscr{H}_2^+$ write $k_0 = k$ and then define recursively, on $x \geqslant 0$,

$$h_{-r}(x) = -e^{-ibx}\left\{\frac{k_{-r+1}(x)}{ib} + \frac{k'_{-r+1}(x)}{(ib)^2} + \ldots + \frac{k^{(s-1)}_{-r+1}(x)}{(ib)^s}\right\} - \frac{1}{(ib)^s}\int_x^\infty k^{(s)}_{-r+1}(t)\,e^{-ibt}\,dt, \tag{16}$$

where $k_{-r}(x) = h_{-r}(x)\,e^{ibx}$ for $r = 1, 2, \ldots$. The integer s is chosen so that $\delta s > N+1$; the infinite integral in (16) then converges absolutely, at any rate when $r = 1$. It will now be proved by induction that this is true for general values of r.

Assume that

$$|k^{(q)}_{-r+1}(x)| < C_q(1+x^2)^{N-\delta q} \tag{17}$$

when $x \geqslant 0$; this certainly holds when $r = 1$ by definition 3. The inequality (17) implies that the integral in (16) converges absolutely and so

$$k'_{-r}(x) = -\frac{k'_{-r+1}(x)}{ib} - \ldots - \frac{k^{(s-1)}_{-r+1}(x)}{(ib)^{s-1}} - \frac{e^{ibx}}{(ib)^{s-1}}\int_x^\infty k^{(s)}_{-r+1}(t)\,e^{-ibt}\,dt. \tag{18}$$

By combining (16) and (18) we obtain

$$k'_{-r}(x) - ibk_{-r}(x) = k_{-r+1}(x). \tag{19}$$

Also, an integration by parts (remembering that $k^{(s)}_{-r+1}$ vanishes at infinity) enables (18) to be written as

$$k'_{-r}(x) = -\frac{k'_{-r+1}(x)}{ib} - \ldots - \frac{k^{(s)}_{-r+1}(x)}{(ib)^s} - \frac{e^{ibx}}{(ib)^s}\int_x^\infty k^{(s+1)}_{-r+1}(t)\,e^{-ibt}\,dt.$$

This formula is the same as that for k_{-r} except that k'_{-r+1} occurs in place of k_{-r+1}. Hence

$$k^{(q)}_{-r}(x) = -\frac{k^{(q)}_{-r+1}(x)}{ib} - \ldots - \frac{k^{(s-1+q)}_{-r+1}(x)}{(ib)^s} - \frac{e^{ibx}}{(ib)^s}\int_x^\infty k^{(s+q)}_{-r+1}(t)\,e^{-ibt}\,dt.$$

It follows from (17) that

$$|k^{(q)}_{-r}(x)| < C_q(1+x^2)^{N-\delta q} + C_{s+q}\int_x^\infty (1+t^2)^{N-\delta(s+q)}\,\mathrm{d}t$$

$$< C_q(1+x^2)^{N-\delta q} + C_{s+q}(1+x^2)^{-\delta q}\int_0^\infty (1+t^2)^{N-\delta s}\,\mathrm{d}t.$$

The integral is finite on account of the choice of s and so, since $N \geqslant 0$,

$$|k^{(q)}_{-r}(x)| < C_q(1+x^2)^{N-\delta q}. \tag{20}$$

This inequality is the same as (17) with r replaced by $r+1$. But (17) is valid for $r = 1$ and so, by induction, holds for every positive integer r. Consequently, the integral in (16) converges absolutely and h_{-r} is well defined.

The preceding results for k_{-r} can be employed to give properties of h_{-r}. For example, (20) implies that

$$|h^{(q)}_{-r}(x)| < C_q(1+x^2)^N, \tag{21}$$

whereas (19) shows that

$$h'_{-r}(x) = h_{-r+1}(x), \quad h'_{-1}(x) = h(x). \tag{22}$$

When $\alpha > 0$, define u_r by $u_r = h_{-r}(\alpha x)/\alpha^r$. Then, from (22),

$$\partial u_r/\partial x = u_{r-1} \tag{23}$$

and repeated application of this result gives $\partial^r u_r/\partial x^r = h(\alpha x)$. Moreover, by using (21), we find

$$\left|\frac{\partial^q u_r}{\partial \alpha^q}\right| < C_q(1+\alpha^2 x^2)^N (1+\alpha^2)^{\frac{1}{2}q}/\alpha^{q+r} \tag{24}$$

which clearly satisfies (2).

Similarly, $v_r = h_{-r}(\alpha x)/x^r$ when $x > 0$ meets the conditions of (3).

It has thus been shown that $h \in \mathscr{H}^+$. It follows from theorem 1 that $\chi(x)\,h(\alpha x) \in \mathscr{H}^+$. Bearing in mind that one possible choice for χ is unity we therefore state

Theorem 5. *If $h \in \mathscr{H}_2^+$ and if χ satisfies the conditions of theorem* 1, *then* $\chi(x)\,h(\alpha x) \in \mathscr{H}^+$.

An analogous theorem can be deduced from theorem 2.

It is also obvious from (20) and the relation $h_{-r}(x)\,\mathrm{e}^{\mathrm{i}bx} = k_{-r}(x)$ that we have

Lemma 3. *If $h \in \mathscr{H}_2^+$ then $h_{-r} \in \mathscr{H}_2^+$ with the same N and δ.*

3. The generalized transforms

The kernel functions $h(\alpha, x)$ introduced in the preceding section provide a basis for defining transforms of generalized functions which are zero for $x < 0$. The class of such generalized functions will be denoted by K_+. It is shown in appendix B that any generalized function in K_+ can be defined by a regular sequence of good functions which vanish identically on $x \leqslant 0$. (The terminology employed is that of Lighthill (1958) and Jones (1966 *b*).) A good function that vanishes identically on $x \leqslant 0$ will be signified by the symbol γ^+, the $+$ sign being attached to make it quite clear that a restricted class of good functions is involved.

Consider

$$\int_0^\infty \gamma^+(x)\,h(\alpha, x)\,\mathrm{d}x$$

when $h \in \mathscr{H}^+$ and $\alpha \geqslant 0$. The lower limit of integration could, if desired, be taken as $-\infty$ by ascribing finite values to h on $x < 0$, since γ^+ is identically zero on $x \leqslant 0$, but this procedure has no special advantage for our purpose. Now

$$\left|\int_0^\infty \gamma^+(x)\,\frac{\partial^q}{\partial\alpha^q}h(\alpha,x)\,\mathrm{d}x\right| < C_{0q}(1+\alpha^2)^{a_{0q}}\int_0^\infty |\gamma^+(x)|\,(1+x^2)^{b_{0q}}\,\mathrm{d}x$$

from (1). The integral on the right is finite because γ^+ is good. Therefore, the integral on the left exists and converges uniformly, with respect to α, for all finite $\alpha \geqslant 0$. Consequently (see, for example, Bartle (1964))

$$\frac{\mathrm{d}^q}{\mathrm{d}\alpha^q}\int_0^\infty \gamma^+(x)\,h(\alpha,x)\,\mathrm{d}x = \int_0^\infty \gamma^+(x)\,\frac{\partial^q}{\partial\alpha^q}h(\alpha,x)\,\mathrm{d}x, \tag{25}$$

a right derivative being employed at $\alpha = 0$. Hence

$$\int_0^\infty \gamma^+(x)\,h(\alpha,x)\,\mathrm{d}x$$

is infinitely differentiable (in the ordinary sense) on $\alpha \geqslant 0$.

Next, observe that on $\alpha > 0$

$$\frac{\partial^{r-p}}{\partial x^{r-p}}\frac{\partial^{p+q}u_r}{\partial x^p\,\partial\alpha^q} = \frac{\partial^q h}{\partial\alpha^q} = \frac{\partial^{r-p}}{\partial x^{r-p}}\frac{\partial^q u_{r-p}}{\partial\alpha^q}$$

for $0 \leqslant p \leqslant r$. Therefore $\partial^{p+q}u_r/\partial x^p\,\partial\alpha^q$ differs from $\partial^q u_{r-p}/\partial\alpha^q$ by at most a polynomial of degree $(r-p-1)$ in x. Taken in conjunction with (1), this shows that $\partial^{p+q}u_r/\partial x^p\,\partial\alpha^q$ is bounded by some power of x as $x\to\infty$. Hence an integration by parts gives

$$\int_0^\infty \gamma^+(x)\,\frac{\partial^q}{\partial\alpha^q}h(\alpha,x)\,\mathrm{d}x = \int_0^\infty \gamma^+(x)\,\frac{\partial^{r+q}u_r}{\partial x^r\,\partial\alpha^q}\,\mathrm{d}x$$
$$= -\int_0^\infty \gamma^{+\prime}(x)\,\frac{\partial^{r+q-1}u_r}{\partial x^{r-1}\,\partial\alpha^q}\,\mathrm{d}x$$

since γ^+ vanishes at the origin and tends to zero faster than any inverse power of x as $x\to\infty$. Since the derivatives of γ^+ also have these properties, repeated integration by parts leads to

$$\int_0^\infty \gamma^+(x)\,\frac{\partial^q}{\partial\alpha^q}h(\alpha,x)\,\mathrm{d}x = (-)^r\int_0^\infty \gamma^{+(r)}(x)\,\frac{\partial^q u_r}{\partial\alpha^q}\,\mathrm{d}x. \tag{26}$$

A use of (2) supplies

$$\left|\alpha^p\int_0^\infty \gamma^+(x)\,\frac{\partial^q}{\partial\alpha^q}h(\alpha,x)\,\mathrm{d}x\right| < C_{qr}\,\frac{(1+\alpha^2)^{M_q+a_1r}}{\alpha^{P_{qr}-p}}\int_0^\infty |\gamma^{+(r)}(x)|\,(1+x^2)^{M_{qr}}\,\mathrm{d}x. \tag{27}$$

on $\alpha > 0$. The integral on the right of (27) is bounded and independent of α. For given p and q r can be chosen so that $2M_q+2a_1r-P_{qr}+p < 0$ since $a_1 < \frac{1}{2}$ and P_{qr} increases with r at least as rapidly as r on account of (4). Such a choice of r ensures that the right-hand side of (27) tends to zero as $\alpha\to\infty$.

This result, taken together with (25), implies that

$$\int_0^\infty \gamma^+(x)\,h(\alpha,x)\,\mathrm{d}x$$

and its derivatives with respect to α tend to zero faster than any inverse power of α as $\alpha\to\infty$. Briefly, it may be said that the integral is good on $\alpha \geqslant 0$.

By interchanging the roles of α and x, and using v_r instead of u_r, a corresponding result for

$$\int_0^\infty \gamma^+(\alpha)\,h(\alpha,x)\,\mathrm{d}\alpha$$

is obtained. Hence we have proved

LEMMA 4. *If $h \in \mathscr{H}^+$ and if γ^+ is a good function which vanishes identically on $x \leqslant 0$, then*

$$\int_0^\infty \gamma^+(x)\,h(\alpha,x)\,\mathrm{d}x$$

is good on $\alpha \geqslant 0$ and

$$\int_0^\infty \gamma^+(\alpha)\,h(\alpha,x)\,\mathrm{d}\alpha$$

is good on $x \geqslant 0$.

Let $\gamma^+(\alpha)$ be any good function which vanishes identically on $\alpha \leqslant 0$. Let $\{\gamma_n^+\}$ be a regular sequence of good functions, each identically zero for $x \leqslant 0$, defining the generalized function $g \in K_+$. Then lemma 4 shows that

$$\left\{\int_0^\infty \gamma_n^+(x)\,h(\alpha,x)\,\mathrm{d}x\right\}$$

is a sequence of functions which are good on $\alpha \geqslant 0$. Also

$$\int_0^\infty \gamma^+(\alpha)\int_0^\infty \gamma_n^+(x)\,h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha = \int_0^\infty \gamma_n^+(x)\int_0^\infty \gamma^+(\alpha)\,h(\alpha,x)\,\mathrm{d}\alpha\,\mathrm{d}x \tag{28}$$

because the integral on the left is absolutely convergent by (1).

Let $\eta_0(x)$ be an infinitely differentiable function such that

$$\begin{aligned}\eta_0(x) &= 1 \quad (x \geqslant 1)\\ &= 0 \quad (x \leqslant \tfrac{1}{2}).\end{aligned}$$

Then

$$\eta_0(x)\int_0^\infty \gamma^+(\alpha)\,h(\alpha,x)\,\mathrm{d}\alpha$$

is a good function of x (including negative values) because, by lemma 4, the integral is good on $x \geqslant 0$. Hence

$$\lim_{n\to\infty}\int_0^\infty \gamma_n^+(x)\,\eta_0(x)\int_0^\infty \gamma^+(\alpha)\,h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha = \int_0^\infty g(x)\,\eta_0(x)\int_0^\infty \gamma^+(\alpha)h(\alpha,x)\,\mathrm{d}\alpha\,\mathrm{d}x. \tag{29}$$

Obviously

$$\int_0^\infty \gamma^+(\alpha)\,h(\alpha,x)\,\mathrm{d}x \in L^r(0,1),$$

as defined in appendix A, for any finite r. In addition, $\{[1-\eta_0(x)]\,\gamma_n^+(x)\}$ is a regular sequence of good functions, zero for $x<0$ and $x>1$, which defines the generalized function $\{1-\eta_0(x)\}g(x)$. Therefore an application of corollary A 1 gives

$$\lim_{n\to\infty}\int_0^1 \{1-\eta_0(x)\}\,\gamma_n^+(x)\int_0^\infty \gamma^+(\alpha)\,h(\alpha,x)\,\mathrm{d}\alpha\,\mathrm{d}x = \int_0^1 \{1-\eta_0(x)\}\,g(x)\int_0^\infty \gamma^+(\alpha)\,h(\alpha,x)\,\mathrm{d}\alpha\,\mathrm{d}x, \tag{30}$$

the (possibly) singular integral on the right being interpreted in the sense of appendix A. The upper limit of unity on the right-hand side can be replaced by ∞ since $1-\eta_0$ and its derivatives vanish identically for $x \geqslant 1$. Then the combination of (28) to (30) leads to

$$\lim_{n\to\infty}\int_0^\infty \gamma^+(\alpha)\int_0^\infty \gamma_n^+(x)\,h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha = \int_0^\infty g(x)\int_0^\infty \gamma^+(\alpha)\,h(\alpha,x)\,\mathrm{d}\alpha\,\mathrm{d}x. \tag{31}$$

Hence
$$\left\{\int_0^\infty \gamma_n^+(x)\, h(\alpha, x)\,\mathrm{d}x\right\}$$
is a sequence of functions which converges in the generalized sense on $\alpha > 0$, and so defines a generalized function on $\alpha > 0$. Accordingly, we have demonstrated

THEOREM 6. *If $h \in \mathscr{H}^+$ and if $g \in K_+$ is defined by the regular sequence $\{\gamma_n^+\}$, then*
$$\left\{\int_0^\infty \gamma_n^+(x)\, h(\alpha, x)\,\mathrm{d}x\right\}$$
defines a generalized function on $\alpha > 0$ which will be denoted by
$$\int_0^\infty g(x)\, h(\alpha, x)\,\mathrm{d}x.$$

It may happen that it can be shown that
$$\lim_{n\to\infty}\int_{-\infty}^\infty \gamma(\alpha)\, H(\alpha)\int_0^\infty \gamma_n^+(x)\, h(\alpha, x)\,\mathrm{d}x\,\mathrm{d}\alpha$$
exists, where γ is any good function and $H(\alpha)$ is the Heaviside unit function which is unity for $\alpha > 0$ and zero for $\alpha < 0$. In that case the sequence would define a generalized function without restriction on α. When this occurs it will be signified by using the notation
$$H(\alpha)\int_0^\infty g(x)\, h(\alpha, x)\,\mathrm{d}x$$
for the generalized transform.

There are several points which stem from theorem 6 and the notation which it introduces. First, $\mathrm{e}^{-\mathrm{i}\alpha x} \in \mathscr{H}_1^+$ and $\mathscr{H}_1^+ \subset \mathscr{H}^+$ so that Fourier transforms of generalized functions in K_+ are covered by the theorem. Since the defining sequence in theorem 6 is then the same as that used in the standard theory of Fourier transforms (see, for example, Jones 1966*b*), theorem 6 agrees with the standard theory on $\alpha > 0$ when $g \in K_+$. This means that the known Fourier transforms for generalized functions in K_+ can be used when $h(\alpha, x) = \mathrm{e}^{-\mathrm{i}\alpha x}$.

Secondly, (31) can be expressed as
$$\int_0^\infty \gamma^+(\alpha)\int_0^\infty g(x)\, h(\alpha, x)\,\mathrm{d}x\,\mathrm{d}\alpha = \int_0^\infty g(x)\int_0^\infty \gamma^+(\alpha)\, h(\alpha, x)\,\mathrm{d}\alpha\,\mathrm{d}x \tag{32}$$
which is analogous to saying that the order of integration can be inverted.

Thirdly, the theorem is in accordance with the customary usage for ordinary functions. Suppose, in fact, that f is an ordinary function, zero for $x < 0$, such that
$$\int_0^\infty (1+x^2)^{b_{00}}\,|f(x)|\,\mathrm{d}x < \infty.$$
Then the argument leading to (28) can be reproduced with f in place of γ_n^+; thus (32) will hold with f substituted for g. On the other hand, such an f is in K_+ and can therefore be defined as a generalized function by an appropriate $\{\gamma_n^+\}$ (see, for example, appendix B); this would also give (32) with f occupying the place of g. Consequently, whether f is thought of as an ordinary function or as a generalized function consistent results are obtained.

Finally, if $g(x) = \delta^{(m)}(x)$ equation (A 4) indicates that
$$\int_0^\infty \delta^{(m)}(x)\int_0^\infty \gamma^+(\alpha)\, h(\alpha, x)\,\mathrm{d}\alpha\,\mathrm{d}x = (-)^m\left[\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_0^\infty \gamma^+(\alpha)\, h(\alpha, x)\,\mathrm{d}\alpha\right]_{x=0}$$
$$= (-)^m\int_0^\infty \gamma^+(\alpha)\left[\frac{\partial^m}{\partial x^m}h(\alpha, x)\right]_{x=0}\mathrm{d}\alpha$$

from (25). It follows from (32) that

$$\int_0^\infty \delta^{(m)}(x)\, h(\alpha, x)\, \mathrm{d}x = (-)^m \left[\frac{\partial^m}{\partial x^m} h(\alpha, x)\right]_{x=0} \tag{33}$$

on $\alpha > 0$.

4. Some properties of the transforms

In deriving certain theorems concerning the transforms it is frequently helpful to take a limit in the generalized sense. To avoid constant repetition of the phrase 'in the generalized sense' the limit in a generalized sense will be denoted by Lim, the notation lim being reserved for the limit taken in the usual way. Thus

$$\operatorname*{Lim}_{n\to\infty} \gamma_n^+ = g \quad (x > 0)$$

is equivalent to the statement

$$\lim_{n\to\infty} \int_0^\infty \gamma^+(x)\, \gamma_n^+(x)\, \mathrm{d}x = \int_0^\infty \gamma^+(x)\, g(x)\, \mathrm{d}x$$

for any good γ^+ which is zero for $x < 0$.

Our first task is to obtain the generalized derivative of the generalized function defined in theorem 6. Now, the generalized derivative can be defined on $\alpha > 0$ by the sequence of derivatives of the terms in

$$\left\{\int_0^\infty \gamma_n^+(x)\, h(\alpha, x)\, \mathrm{d}x\right\}.$$

In other words, the generalized derivative

$$\frac{\mathrm{d}^q}{\mathrm{d}\alpha^q}\int_0^\infty g(x)\, h(\alpha, x)\, \mathrm{d}x = \operatorname*{Lim}_{n\to\infty} \frac{\mathrm{d}^q}{\mathrm{d}\alpha^q}\int_0^\infty \gamma_n^+(x)\, h(\alpha, x)\, \mathrm{d}x \tag{34}$$

on $\alpha > 0$. It follows from (25) that

$$\operatorname*{Lim}_{n\to\infty} \int_0^\infty \gamma_n^+(x)\, \frac{\partial^q}{\partial \alpha^q} h(\alpha, x)\, \mathrm{d}x$$

is a generalized function of α on $\alpha > 0$; it may be conveniently denoted by

$$\int_0^\infty g(x)\, \frac{\partial^q}{\partial \alpha^q} h(\alpha, x)\, \mathrm{d}x.$$

Combining this with (34) we obtain

Theorem 7. *If $h \in \mathscr{H}^+$ and $g \in K_+$, then*

$$\frac{\mathrm{d}^q}{\mathrm{d}\alpha^q}\int_0^\infty g(x)\, h(\alpha, x)\, \mathrm{d}x = \int_0^\infty g(x)\, \frac{\partial^q}{\partial \alpha^q} h(\alpha, x)\, \mathrm{d}x$$

on $\alpha > 0$.

The derivative on the left-hand side is a generalized one; that on the right is a partial derivative in the conventional sense. The theorem is consistent with standard results for ordinary functions which have the requisite properties.

Next, observe that systematic integration by parts gives

$$\operatorname*{Lim}_{n\to\infty} \int_0^\infty \gamma_n^+(x)\, \frac{\partial^q}{\partial x^q} h(\alpha, x)\, \mathrm{d}x = \operatorname*{Lim}_{n\to\infty} (-)^q \int_0^\infty \gamma_n^{+(q)}(x)\, h(\alpha, x)\, \mathrm{d}x$$

because the behaviour of γ_n^+ and its derivatives at the end-points ensures that there is no contribution from the limits of integration. The right-hand side exists by theorem 6 and the limit process replaces $\gamma_n^{+(q)}$ by $g^{(q)}$. Therefore the left-hand side exists as a generalized function, which may be denoted by

$$\int_0^\infty g(x)\,\frac{\partial^q}{\partial x^q}h(\alpha, x)\,\mathrm{d}x.$$

The generalized limit of (26) then attaches a significance to

$$\int_0^\infty g^{(r)}\,\partial^q u_r/\partial x^q\,\mathrm{d}x$$

and we have

THEOREM 8. *If $h \in \mathscr{H}^+$ and $g \in K_+$, then*

$$\int_0^\infty g(x)\,\frac{\partial^q}{\partial x^q}h(\alpha, x)\,\mathrm{d}x = (-)^q\int_0^\infty g^{(q)}(x)\,h(\alpha, x)\,\mathrm{d}x$$
$$= (-)^r\int_0^\infty g^{(r)}(x)\,\frac{\partial^q u_r}{\partial x^q}\,\mathrm{d}x$$

on $\alpha > 0$.

It must be remembered in the application of this theorem that $g^{(q)}$ is the generalized derivative as calculated for K_+ and not as calculated on $x > 0$. Thus

$$\int_0^\infty (\partial h/\partial x)\,\mathrm{d}x = -h(\alpha, 0) \quad (\alpha > 0) \tag{35}$$

because, in this case, $g(x) = H(x)$ and so $g'(x) = \delta(x)$ (not zero) in theorem 8 from which (35) follows through (33).

A corresponding theorem involving v_r can be derived. Because of (3) and the vanishing of $\gamma^+(x)$ at the origin faster than any power of x the product of γ^+ and a derivative of v_s can be defined to be continuous at $x = 0$. Hence

$$\operatorname*{Lim}_{n\to\infty}\frac{\mathrm{d}^s}{\mathrm{d}\alpha^s}\int_0^\infty \gamma_n^+(x)\,\frac{\partial^q v_s}{\partial x^q}\,\mathrm{d}x = \operatorname*{Lim}_{n\to\infty}\int_0^\infty \gamma_n^+(x)\,\frac{\partial^q}{\partial x^q}h(\alpha, x)\,\mathrm{d}x$$
$$= \int_0^\infty g(x)\,\frac{\partial^q}{\partial x^q}h(\alpha, x)\,\mathrm{d}x$$

from the analysis above. The generalized function defined by the left-hand side merely replaces γ_n^+ by g; this is consistent with theorem 7. Hence

THEOREM 9. *If $h \in \mathscr{H}^+$ and $g \in K_+$, then*

$$\frac{\mathrm{d}^s}{\mathrm{d}\alpha^s}\int_0^\infty g(x)\,\frac{\partial^q v_s}{\partial x^q}\,\mathrm{d}x = \int_0^\infty g(x)\,\frac{\partial^q}{\partial x^q}h(\alpha, x)\,\mathrm{d}x$$

on $\alpha > 0$.

The remainder of this section is concerned with linear changes of argument. Let σ and τ be real numbers with σ positive. Write

$$\gamma_n^+(\sigma x + \tau) = \gamma_n(x);$$

then γ_n is a good function which vanishes identically for $x \leqslant -\tau/\sigma$ and the sequence $\{\gamma_n\}$ defines a generalized function $g_0(x)$ which is zero for $x < -\tau/\sigma$. Also, the substitution $\sigma y + \tau = x$ gives

$$\operatorname*{Lim}_{n\to\infty}\int_{-\tau/\sigma}^\infty \gamma_n(y)\,h(\alpha, \sigma y + \tau)\,\mathrm{d}y = \operatorname*{Lim}_{n\to\infty}\int_0^\infty \gamma_n^+(x)\,h(\alpha, x)\,\mathrm{d}x/\sigma$$
$$= \int_0^\infty g(x)\,h(\alpha, x)\,\mathrm{d}x/\sigma. \tag{36}$$

Hence, the left-hand side defines a generalized function and we agree to write

$$\lim_{n\to\infty}\int_{-\tau/\sigma}^{\infty}\gamma_n(y)\,h(\alpha,\sigma y+\tau)\,\mathrm{d}y=\int_{-\tau/\sigma}^{\infty}g_0(y)\,h(\alpha,\sigma y+\tau)\,\mathrm{d}y \tag{37}$$

which is not in conflict with earlier notation. Since

$$g_0(y)=\lim_{n\to\infty}\gamma_n(y)=\lim_{n\to\infty}\gamma_n^+(\sigma y+\tau)=g(\sigma y+\tau)$$

(36) and (37) provide

THEOREM 10. *If $h\in\mathscr{H}^+$ and g_0 is zero for $x<-\tau/\sigma$, then*

$$\int_{-\tau/\sigma}^{\infty}g_0(x)\,h(\alpha,\sigma x+\tau)\,\mathrm{d}x=\int_0^{\infty}g_0\left(\frac{x-\tau}{\sigma}\right)h(\alpha,x)\,\frac{\mathrm{d}x}{\sigma}$$

on $\alpha>0$, when σ and τ are real with $\sigma>0$.

An obvious variation on this theorem is that, if g_1 is zero for $x>\tau/\sigma$,

$$\int_{-\infty}^{\tau/\sigma}g_1(x)\,h(\alpha,\tau-\sigma x)\,\mathrm{d}x=\int_0^{\infty}g_1\left(\frac{\tau-x}{\sigma}\right)h(\alpha,x)\,\frac{\mathrm{d}x}{\sigma} \tag{38}$$

when $\sigma>0$. Putting $\sigma=-\sigma'$ we obtain

$$\int_{-\infty}^{-\tau/\sigma'}g_1(x)\,h(\alpha,\sigma' x+\tau)\,\mathrm{d}x=-\int_0^{\infty}g_1\left(\frac{x-\tau}{\sigma'}\right)h(\alpha,x)\,\frac{\mathrm{d}x}{\sigma'} \tag{39}$$

when $\sigma'<0$.

5. THE STRUCTURE OF A GENERALIZED TRANSFORM

A generalized transform has been defined for $\alpha>0$. The object of this section is to derive a generalized function which is zero on $\alpha<0$ and which agrees with the transform on $\alpha>0$. In the process something of the structure of a generalized transform will be elucidated.

When $g\in K_+$ it is possible to write $g=f^{(r)}$ for some finite r, where the continuous function f is zero on $x<0$ and is bounded by a polynomial as $x\to\infty$. It follows that $f\in K_+$. Therefore, by theorem 8,

$$\int_0^{\infty}g(x)\,h(\alpha,x)\,\mathrm{d}x=(-)^r\int_0^{\infty}f(x)\,\frac{\partial^r}{\partial x^r}h(\alpha,x)\,\mathrm{d}x$$

on $\alpha>0$. Now

$$\int_0^{\infty}\{1-\eta_0(x)\}f(x)\,\frac{\partial^r}{\partial x^r}h(\alpha,x)\,\mathrm{d}x=\int_0^{1}\{1-\eta_0(x)\}f(x)\,\frac{\partial^r}{\partial x^r}h(\alpha,x)\,\mathrm{d}x$$

so that, on account of (1), the left-hand side is an ordinary function of α which is continuous on $\alpha\geqslant 0$ and which is bounded by a polynomial as $\alpha\to\infty$. Hence, for any given non-negative integer s, there is a continuous f_1, bounded by a polynomial, such that

$$\frac{\mathrm{d}^s f_1}{\mathrm{d}\alpha^s}=\int_0^{\infty}\{1-\eta_0(x)\}f(x)\,\frac{\partial^r}{\partial x^r}h(\alpha,x)\,\mathrm{d}x.$$

It can be concluded that the generalized transform of any generalized function which vanishes outside a finite interval is a continuous function on $\alpha\geqslant 0$.

In addition, by theorem 9,

$$\int_0^{\infty}\eta_0(x)f(x)\,\frac{\partial^r}{\partial x^r}h(\alpha,x)\,\mathrm{d}x=\frac{\mathrm{d}^s}{\mathrm{d}\alpha^s}\int_0^{\infty}\eta_0(x)f(x)\,\frac{\partial^r v_s}{\partial x^r}\,\mathrm{d}x. \tag{40}$$

The integrand on the right is continuous, being zero for $x \leqslant \frac{1}{2}$, and s can be chosen so large that the integral converges absolutely by virtue of (3). Therefore the right-hand side can be expressed as $\mathrm{d}^s f_2/\mathrm{d}\alpha^s$ where f_2 is continuous on $\alpha \geqslant 0$ and bounded by a polynomial. Hence

$$\int_0^\infty g(x)\, h(\alpha, x)\, \mathrm{d}x = (-)^r \frac{\mathrm{d}^s}{\mathrm{d}\alpha^s}(f_1+f_2) \tag{41}$$

on $\alpha > 0$.

Since $\{f_1(\alpha)+f_2(\alpha)\}H(\alpha)$ is a well-defined function in K_+, the generalized derivative

$$(-)^r \frac{\mathrm{d}^s}{\mathrm{d}\alpha^s}\{(f_1+f_2)\, H\} = \hat{g} \tag{42}$$

is a generalized function $\hat{g}$ in K_+ which agrees with the right-hand side of (4) on $\alpha > 0$. It has thus been established that there is a $\hat{g} \in K_+$ such that

$$\hat{g}(\alpha) = \int_0^\infty g(x)\, h(\alpha, x)\, \mathrm{d}x \quad (\alpha > 0), \tag{43}$$

when $h \in \mathscr{H}^+$. The form of $\hat{g}$ is given by (42) but the representation is not unique because a generalized function such as

$$\sum_{m=0}^{M} a_m\, \delta^{(m)}(\alpha)$$

could be added to $\hat{g}$ without destroying the relation (43).

If g vanishes outside a finite interval, say (a, b), we can multiply g by a fine function which is unity on (a, b) without affecting the generalized transform. It is then evident from the above analysis that $\hat{g}$ will contain only a term of the type f_1. In other words, when g vanishes outside a finite interval, the generalized transform is a continuous function of α on $\alpha \geqslant 0$.

6. Limits

A theorem of some value in calculating generalized transforms is

Theorem 11. *If* $h \in \mathscr{H}^+$ *and if* $\operatorname*{Lim}_{m\to\infty} g_m = g$ *where* $g_m \in K_+$ *and* $g \in K_+$, *then*

$$\operatorname*{Lim}_{m\to\infty} \int_0^\infty g_m(x)\, h(\alpha, x)\, \mathrm{d}x = \int_0^\infty g(x)\, h(\alpha, x)\, \mathrm{d}x$$

on $\alpha > 0$.

It is important to note that the requirement $\operatorname*{Lim}_{m\to\infty} g_m = g$ must hold for all x, i.e. more is being asked than that the generalized limit be g on $x > 0$.

Proof. From (32)

$$\int_0^\infty \gamma^+(\alpha) \int_0^\infty g_m(x)\, h(\alpha, x)\, \mathrm{d}x\, \mathrm{d}\alpha = \int_0^\infty g_m(x) \int_0^\infty \gamma^+(\alpha) h(\alpha, x)\, \mathrm{d}\alpha\, \mathrm{d}x.$$

According to lemma 4 the inner integral on the right is good on $x \geqslant 0$. If it is handled in the same way as in the proof of theorem 6, an application of corollary A 1 gives

$$\lim_{m\to\infty} \int_0^\infty g_m(x) \int_0^\infty \gamma^+(\alpha)\, h(\alpha, x)\, \mathrm{d}\alpha\, \mathrm{d}x = \int_0^\infty g(x) \int_0^\infty \gamma^+(\alpha)\, h(\alpha, x)\, \mathrm{d}\alpha\, \mathrm{d}x$$

and the theorem is proved by using (32) on the right-hand side.

If $g = \sum_{n=1}^{\infty} g_n$, then $g = \operatorname{Lim}_{m\to\infty} \sum_{n=1}^{m} g_n$ and so an immediate consequence of theorem 11 is

COROLLARY 11*a*. *If* $h \in \mathscr{H}^+$ *and if* $g = \sum_{n=1}^{\infty} g_n$ *where* $g_n \in K_+$, *then*

$$\int_0^\infty g(x)\, h(\alpha, x)\, \mathrm{d}x = \sum_{n=1}^{\infty} \int_0^\infty g_n(x)\, h(\alpha, x)\, \mathrm{d}x$$

on $\alpha > 0$.

Now let $\chi_\mu(x)$ be infinitely differentiable (in the ordinary sense) on $x \geqslant 0$ and be such that

$$|\chi_\mu(x) - 1| \leqslant \epsilon_0 (1+x^2)^{p_0}, \quad \left|\frac{\partial^r \chi_\mu}{\partial x^r}\right| \leqslant \epsilon_r (1+x^2)^{p_r} \quad (r = 1, 2, \ldots) \tag{44}$$

on $x \geqslant 0$, where $\epsilon_s \to 0$ uniformly as $\mu \to +0$ for $s = 0, 1, \ldots$. For example, one could take $\chi_\mu(x) = \mathrm{e}^{-\mu x}$. Then, if γ is any good function and $g \in K_+$,

$$\int_{-\infty}^{\infty} \gamma(x)\, \chi_\mu(x)\, g(x)\, \mathrm{d}x = (-)^r \int_0^\infty f(x) \frac{\partial^r}{\partial x^r} \{\gamma(x)\, \chi_\mu(x)\}\, \mathrm{d}x$$

on using the representation $g = f^{(r)}$ where $f \in K_+$, is continuous and bounded by a polynomial. On account of (44), the right-hand side tends to

$$(-)^r \int_0^\infty f(x) \frac{\partial^r \gamma}{\partial x^r}\, \mathrm{d}x = \int_{-\infty}^{\infty} \gamma(x)\, g(x)\, \mathrm{d}x$$

as $\mu \to +0$. Hence $\operatorname{Lim}_{\mu\to+0} \chi_\mu g = g$ without restriction on x. Taken in conjunction with theorem 11, this gives

COROLLARY 11*b*. *If* $h \in \mathscr{H}^+$, $g \in K_+$ *and* χ_μ *satisfies the conditions stated in* (44), *then*

$$\operatorname{Lim}_{\mu\to+0} \int_0^\infty \chi_\mu(x)\, g(x)\, h(\alpha, x)\, \mathrm{d}x = \int_0^\infty g(x)\, h(\alpha, x)\, \mathrm{d}x$$

on $\alpha > 0$.

7. Inversion theorems

The primary concern of this paper is the asymptotic behaviour of generalized transforms, but inversion theorems are frequently of great help in solving problems in applied mathematics. Therefore we shall indicate some of the conditions under which an inversion theorem will hold, and also display some of the likely causes of failure of such a theorem.

If an inversion theorem is to be valid for a class of generalized functions, we certainly expect that it will apply to functions of the type γ^+. Therefore, suppose that $h_1 \in \mathscr{H}^+$ is such that

$$\int_0^\infty h_1(y, \alpha) \int_0^\infty \gamma^+(x)\, h(\alpha, x)\, \mathrm{d}x\, \mathrm{d}\alpha = \gamma^+(y) \tag{45}$$

on $y > 0$ for every good $\gamma^+(x)$ which vanishes identically on $x \leqslant 0$. The double integral exists, although it may not be absolutely convergent, because the inner integral is good on $\alpha \geqslant 0$ by lemma 4. If γ^+ is replaced by γ_n^+, where $\{\gamma_n^+\}$ is a defining sequence for g, and the generalized limit taken,

$$\operatorname{Lim}_{n\to\infty} \int_0^\infty h_1(y, \alpha) \int_0^\infty \gamma_n^+(x)\, h(\alpha, x)\, \mathrm{d}x\, \mathrm{d}\alpha = g(y)$$

on $y > 0$ is obtained.

It is tempting to employ theorem 11 on the left-hand side and replace γ_n^+ by g in the limit. However, theorem 6 only proves that

$$\operatorname*{Lim}_{n\to\infty}\int_0^\infty \gamma_n^+(x)\,h(\alpha,x)\,\mathrm{d}x = \int_0^\infty g(x)\,h(\alpha,x)\,\mathrm{d}x$$

on $\alpha > 0$; for theorem 11 to apply this has to be true in K_+. Consequently, further assumptions must be made if theorem 11 is to be used. For example, if

$$\operatorname*{Lim}_{n\to\infty}\int_0^\infty \gamma_n^+(x)\,h(\alpha,x)\,\mathrm{d}x = \hat{g}(\alpha)$$

where $\hat{g}$ satisfies (43) we can take advantage of theorem 11.

At the moment all that can be stated is

LEMMA 5. *If* (45) *holds with* $h_1,\ h\in\mathscr{H}^+$ *then*

$$\operatorname*{Lim}_{n\to\infty}\int_0^\infty h_1(y,\alpha)\int_0^\infty \gamma_n^+(x)\,h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha = g(y)$$

on $y > 0$, *and if* $$\operatorname*{Lim}_{n\to\infty} H(\alpha)\int_0^\infty \gamma_n^+(x)\,h(\alpha,x)\,\mathrm{d}x = H(\alpha)\int_0^\infty g(x)\,h(\alpha,x)\,\mathrm{d}x$$

then $$\int_0^\infty h_1(y,\alpha)\,H(\alpha)\int_0^\infty g(x)\,h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha = g(y)$$

on $y > 0$.

It is clear from this lemma that an inversion theorem will hold only for certain classes of generalized functions and that these classes will be different in general for different h. However, there are some generalized functions for which inversion always holds, if it is valid at all, and these will now be discussed.

It was pointed out at the end of §5 that the generalized transform is a continuous function when g vanishes outside a finite interval. In that case the second condition of lemma 5 must be met and so we have

THEOREM 12. *If* (45) *holds with* $h_1,\ h\in\mathscr{H}^+$, *if* $g\in K_+$ *and vanishes outside a finite interval, then*

$$\int_0^\infty h_1(y,\alpha)\int_0^\infty g(x)\,h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha = g(y)$$

on $y > 0$.

The factor H has been omitted since it serves no useful purpose in this theorem.

It follows from (33) and theorem 12 that

$$\int_0^\infty h_1(y,\alpha)\left[\frac{\partial^m}{\partial x^m}h(\alpha,x)\right]_{x=0}\mathrm{d}\alpha = 0 \tag{46}$$

on $y > 0$. An immediate deduction is that the integral equation

$$\int_0^\infty h_1(y,\alpha)g_1(\alpha)\,\mathrm{d}\alpha = g_2(y) \quad (y>0)$$

does not have a unique solution unless h and all its derivatives with respect to x are zero at $x = 0$. On the other hand, if $h(\alpha, 0) = 0$, there could not be an inversion theorem without the restriction $y > 0$ otherwise the assertion

$$\int_0^\infty 0\,.\,h_1(y,\alpha)\,\mathrm{d}x = \delta(y)$$

would have to be made and this is manifestly untrue.

There are corresponding results with h and h_1 interchanged. For (45) implies that

$$\int_0^\infty \gamma_1^+(y) \int_0^\infty h_1(y,\alpha) \int_0^\infty \gamma^+(x)\, h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha\,\mathrm{d}y = \int_0^\infty \gamma_1^+(y)\,\gamma^+(y)\,\mathrm{d}y$$

for any γ_1^+. The left-hand side can be written as

$$\int_0^\infty \int_0^\infty \gamma^+(x)\, h(\alpha,x)\,\mathrm{d}x \int_0^\infty \gamma_1^+(y)\, h_1(y,\alpha)\,\mathrm{d}y\,\mathrm{d}\alpha$$

by (32), or as

$$\int_0^\infty \gamma^+(x) \int_0^\infty h(\alpha,x) \int_0^\infty \gamma_1^+(y)\, h_1(y,\alpha)\,\mathrm{d}y\,\mathrm{d}\alpha\,\mathrm{d}x$$

after another application of (32). Hence

$$\int_0^\infty h(\alpha,x) \int_0^\infty \gamma_1^+(y)\, h_1(y,\alpha)\,\mathrm{d}y\,\mathrm{d}\alpha = \gamma_1^+(x) \tag{47}$$

on $x > 0$.

There are analogues to lemma 5 and theorem 12 which arise from (47). For example, if h is a function of αx,

$$\int_0^\infty h(\alpha x)\, h_1(0,\alpha)\,\mathrm{d}\alpha = 0 \quad (x > 0)$$

which shows, when $h_1(0,\alpha) \not\equiv 0$, that there could not possibly be an inversion theorem if the behaviour of the generalized transform at the origin were excluded.

Actually, theorem 12 can be extended to a wider class of generalized functions. Let $\gamma(x)$ be a good function of x and let $\Gamma(\alpha)$ be good on $\alpha \geqslant 0$. Then

$$\int_0^\infty \Gamma(\alpha) \int_0^\infty \gamma_n^+(x)\,\gamma(x)\, h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha = \int_0^\infty \gamma_n^+(x)\,\gamma(x) \int_0^\infty \Gamma(\alpha)\, h(\alpha,x)\,\mathrm{d}\alpha\,\mathrm{d}x$$

by absolute convergence. The inner integral on the right is a continuous differentiable function of x on $x \geqslant 0$ and is $O\{(1+x^2)^{b_{00}}\}$. Hence, when multiplied by $\gamma(x)$, it is good on $x \geqslant 0$ and so

$$\lim_{n\to\infty} \int_0^\infty \gamma_n^+(x)\,\gamma(x) \int_0^\infty \Gamma(\alpha)\, h(\alpha,x)\,\mathrm{d}\alpha\,\mathrm{d}x = \int_0^\infty g(x)\,\gamma(x) \int_0^\infty \Gamma(\alpha)\, h(\alpha,x)\,\mathrm{d}\alpha\,\mathrm{d}x.$$

Therefore

$$\operatorname*{Lim}_{n\to\infty} H(\alpha) \int_0^\infty \gamma_n^+(x)\,\gamma(x)\, h(\alpha,x)\,\mathrm{d}x$$

is a generalized function, without restriction on α, and we have

$$\int_{-\infty}^\infty \Gamma(\alpha)\, H(\alpha) \int_0^\infty g(x)\,\gamma(x)\, h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha = \int_0^\infty g(x)\,\gamma(x) \int_0^\infty \Gamma(\alpha)\, h(\alpha,x)\,\mathrm{d}\alpha\,\mathrm{d}x. \tag{48}$$

Thus lemma 5 is satisfied and inversion holds.

Since γ could be chosen to be zero outside a finite interval, this result includes theorem 12 and is more comprehensive than it. Another choice for $\gamma(x)$ is $\mathrm{e}^{-\mu x}$ $(x > 0)$ and then, from the proof of corollary 11*b*, we have

THEOREM 13. *If* (45) *holds with* $h_1,\, h \in \mathscr{H}^+$ *and if* $g \in K_+$, *then*

$$g(y) = \operatorname*{Lim}_{\mu\to+0} g(y)\,\mathrm{e}^{-\mu y} = \operatorname*{Lim}_{\mu\to+0} \int_0^\infty h_1(y,\alpha)\, H(\alpha) \int_0^\infty g(x)\,\mathrm{e}^{-\mu x}\, h(\alpha,x)\,\mathrm{d}x\,\mathrm{d}\alpha \tag{49}$$

on $y > 0$.

For many purposes theorem 13 will be sufficiently general and so the problem of inversion will not be considered further.

8. The analogue of the Riemann–Lebesgue lemma

As a start to the discussion of the asymptotic behaviour of a generalized transform we shall derive suitable analogues of the Riemann–Lebesgue lemma as it occurs in the theory of Fourier transforms. The analysis of this section will be confined to ordinary functions, but the interval of integration will not necessarily be restricted to $x \geqslant 0$.

Let $h_0(\alpha, x)$, $h_{-1}(\alpha, x)$ be defined on the interval of integration (x is the variable of integration) and be such that

$$\partial h_{-1}(\alpha, x)/\partial x = h_0(\alpha, x), \tag{50}$$

$$|h_0(\alpha, x)| < C\alpha^{2P}(1+x^2)^S \tag{51}$$

for $\alpha \geqslant 1$ and

$$|h_{-1}(\alpha, x)| = o\{\alpha^{2P}(1+x^2)^{S_1}\} \tag{52}$$

as $\alpha \to +\infty$. When these conditions are satisfied on the interval of integration (a, b) we shall write $h_0 \in P(a, b)$; it is not necessary for P to be positive. It is assumed that h_0 is integrable and that h_{-1} is absolutely continuous but no other conditions are imposed in this section. This freedom from restriction will permit the asymptotic estimation in later sections of integrals involving $K(x)\,h(\alpha, x)$ where K may not be infinitely differentiable.

Suppose that f is absolutely continuous on the finite interval $[a, b]$. Then, by integration by parts,

$$\int_a^b f(x)\,h_0(\alpha, x)\,dx = [f(x)\,h_{-1}(\alpha, x)]_a^b - \int_a^b f'(x)\,h_{-1}(\alpha, x)\,dx. \tag{53}$$

Since f is bounded, (52) implies that the first term on the right of (53) is $o(\alpha^{2P})$ as $\alpha \to +\infty$. Also the second term is

$$o\left\{\alpha^{2P}\int_a^b |f'(x)|\,(1+x^2)^{S_1}\,dx\right\} = o(\alpha^{2P}).$$

Therefore, as $\alpha \to +\infty$, the right-hand side of (53) is $o(\alpha^{2P})$.

If (52) were replaced by

$$|h_{-1}(\alpha, x)| = O\{(1+\alpha^2)^{P-\frac{1}{2}\beta}\,(1+x^2)^{S_1}\} \quad (\beta > 0) \tag{54}$$

the same argument shows that the right-hand side of (53) is $O(\alpha^{2P-\beta})$. Consequently, we can state

Lemma 6. *If f is absolutely continuous on the finite interval $[a, b]$ and $h_0 \in P(a, b)$, then*

$$\int_a^b f(x)\,h_0(\alpha, x)\,dx = o(\alpha^{2P})$$

as $\alpha \to +\infty$. If (54) holds

$$\int_a^b f(x)\,h_0(\alpha, x)\,dx = O(\alpha^{2P-\beta}).$$

Next, let $f \in L_1(a, b)$ where, as usual, $f \in L_p(a, b)$ signifies that $|f|^p$ is integrable over (a, b). Then, given $\epsilon > 0$, there is an absolutely continuous κ such that

$$\int_a^b |f(x) - \kappa(x)|\,dx < \epsilon.$$

Hence, by (51),

$$\left|\int_a^b \{f(x) - \kappa(x)\}\,h(\alpha, x)\,dx\right| < C\epsilon\alpha^{2P}$$

since the interval of integration is finite. It follows from lemma 6 that

$$\left|\int_a^b f(x)\,h_0(\alpha, x)\,dx\right| < C\epsilon\alpha^{2P} + o(\alpha^{2P}).$$

Since ϵ can be chosen arbitrarily small we have

LEMMA 7. *If* $f \in L_1(a, b)$, *with* (a, b) *finite, and* $h_0 \in P(a, b)$, *then*

$$\int_a^b f(x)\, h_0(\alpha, x)\, \mathrm{d}x = o(\alpha^{2P})$$

as $\alpha \to +\infty$.

If $f \in L_1(a, b)$ and K is continuous, then $fK \in L_1(a, b)$ and so lemma 7 gives

$$\int_a^b f(x)\, K(x)\, h_0(\alpha, x)\, \mathrm{d}x = o(\alpha^{2P}). \tag{55}$$

This may also be interpreted as saying that h_0 may be replaced in lemma 7 by $K(x)\, h_0(\alpha, x)$ where K is continuous on the interval of integration.

Turning now to infinite intervals of integration we assume that $(1+x^2)^S f(x) \in L_1(a, \infty)$. Then b can be chosen so large that

$$\int_b^\infty (1+x^2)^S\, |f(x)|\, \mathrm{d}x < \epsilon.$$

Hence

$$\left|\int_b^\infty f(x)\, h_0(\alpha, x)\, \mathrm{d}x\right| < C\epsilon\alpha^{2P}$$

from (51). Combining this with lemma 7 we obtain

THEOREM 14. *If* $(1+x^2)^S f(x) \in L_1(a, \infty)$ *and* $h_0 \in P(a, \infty)$, *then*

$$\int_a^\infty f(x)\, h_0(\alpha, x)\, \mathrm{d}x = o(\alpha^{2P})$$

as $\alpha \to +\infty$.

The interval $(-\infty, a)$ could be treated in the same way as (b, ∞) so that there are corresponding theorems for

$$\int_{-\infty}^b \quad \text{and} \quad \int_{-\infty}^\infty.$$

We state only

COROLLARY 14. *Theorem* 14 *remains true if a is replaced by* $-\infty$.

If the assumptions (50) to (52) are met when $\alpha \to -\infty$, then the subsequent lemmas and theorem are still true as $\alpha \to -\infty$ provided that $o(|\alpha|^{2P})$ is used in place of $o(\alpha^{2P})$.

There are several cases of interest in which $h_0(\alpha, x)$ takes the form $h_0(\alpha x)$. In such cases it is worth remarking that the information that $h_0(x) = O\{(1+x^2)^{-M}\}$ with $M > 0$ is not sufficient to ensure that (51) holds for $h_0(\alpha x)$ with a negative P. Thus the smallest value of P which would apply in theorem 14 would be zero. Improvement of this result is not possible as can be seen from the example

$$\int_0^\infty \frac{1}{x^\beta} \frac{\mathrm{e}^{-\mathrm{i}\alpha x}}{(1+\alpha^2x^2)^M}\, \mathrm{d}x = \alpha^{\beta-1} \int_0^\infty \frac{1}{x^\beta} \frac{\mathrm{e}^{-\mathrm{i}x}}{(1+x^2)^M}\, \mathrm{d}x$$

with $M \geqslant 1$. Here the α-dependence does not involve M and, moreover, can be made to dominate any inverse power of α by selecting β sufficiently close to 1.

By assuming slightly more about the integrand it is possible to achieve appreciably better results than lemma 7. We shall consider only the finite interval $(0, c)$ with $c > 0$ but there is no loss in so doing because the theory is clearly applicable to any finite interval after a suitable translation. The notation $f \in \hat{L}_p(0, c)$ will be used to indicate that on the interval $f = \psi_1 - \psi_2$ where $\psi_1, \psi_2 \in L_p(0, c)$, are non-increasing and are bounded at c. By the addition of the same constant to each of ψ_1 and ψ_2 it can be ensured that both ψ_1 and ψ_2 are positive on the interval. It will be assumed that this has been done so that ψ_1 and ψ_2 will be taken as positive non-increasing and bounded at c, though not necessarily at the origin.

In the following ψ will represent either ψ_1 or ψ_2. It will be shown firstly that, when $1 \leqslant p < \infty$,

$$\psi(x) = o(x^{-1/p}) \tag{56}$$

as $x \to +0$. Suppose, in fact, that (56) is not true. Then there is a sequence $\{x_n\}$, with $x_{n+1} < x_n$ and $\lim_{n\to\infty} x_n = 0$, such that

$$\psi(x_n) > Cx_n^{-1/p}$$

for some positive C independent of n. Since ψ is non-increasing

$$\psi(x) > Cx_n^{-1/p}$$

for $x_{n+1} < x \leqslant x_n$. Hence, taking $x_1 \leqslant c$, we have

$$\int_0^{x_1} |\psi(x)|^p \,\mathrm{d}x > \sum_{n=1}^{\infty} (x_n - x_{n+1}) \frac{C^p}{x_n} = C^p \sum_{n=1}^{\infty} \epsilon_n,$$

where $\epsilon_n = 1 - x_{n+1}/x_n$. Each ϵ_n is positive and less than unity; the infinite series converges because the integral is finite since $\psi \in L_p(0, c)$. Hence

$$\sum_{n=1}^{\infty} \ln(1-\epsilon_n)$$

is convergent and so

$$\lim_{n\to\infty} \prod_{m=1}^{n} (1-\epsilon_m)$$

is finite and non-zero. But

$$\prod_{m=1}^{n} (1-\epsilon_m) = x_{n+1}/x_1$$

which tends to zero as $n \to \infty$. Thus a contradiction has been reached and we are forced to conclude that (56) holds.

Having established (56) we now assume that α is so large that $1/\alpha^\beta < c$, where β is the same as in (54). Then, by the second mean value theorem (see, for example, Jones (1966*b*)),

$$\int_{1/\alpha^\beta}^{c} \psi(t)\, h_0(\alpha, t)\,\mathrm{d}t = \psi\left(\frac{1}{\alpha^\beta}+0\right) \int_{1/\alpha^\beta}^{\xi} h_0(\alpha, t)\,\mathrm{d}t$$
$$= \psi\left(\frac{1}{\alpha^\beta}+0\right)\left\{h_{-1}(\alpha, \xi) - h_{-1}\left(\alpha, \frac{1}{\alpha^\beta}\right)\right\},$$

where $1/\alpha^\beta < \xi < c$. It follows from (54) and (56) that

$$\left|\int_{1/\alpha^\beta}^{c} \psi(t)\, h_0(\alpha, t)\,\mathrm{d}t\right| = o(\alpha^{2P-\beta+\beta/p}). \tag{57}$$

Also, from (51)

$$\left|\int_0^{1/\alpha^\beta} \psi(t)\, h_0(\alpha, t)\,\mathrm{d}t\right| < C\alpha^{2P} \int_0^{1/\alpha^\beta} \psi(t)\,\mathrm{d}t$$
$$< o(\alpha^{2P-\beta+\beta/p}) \tag{58}$$

from (56) when $1 < p < \infty$ and from a well-known theorem of integration when $p = 1$. The estimates (57) and (58) combine as

$$\int_0^c \psi(t)\, h_0(\alpha, t)\,\mathrm{d}t = o(\alpha^{2P-\beta+\beta/p}).$$

In view of the definition of f there follows

THEOREM 15. *If* (50), (51) *and* (54) *hold and if* $f \in \hat{L}_p(0, c)$ $(1 \leqslant p < \infty)$, *then*

$$\int_0^c f(t)\, h_0(\alpha, t)\, \mathrm{d}t = o(\alpha^{2P-\beta+\beta/p})$$

as $\alpha \to \infty$.

This is clearly an improvement on lemma 7.

The situation when $p = \infty$ is somewhat different. If $\psi \in L_\infty(0, c)$, it is essentially bounded, i.e. it is bounded except on a set of measure zero. But, $\psi(t_0) > C$ implies that $\psi(t) > C$ for $t \leqslant t_0$ and therefore ψ must, in fact, be bounded except possibly at the origin. Redefinition of ψ at the origin, if necessary, makes ψ bounded. Thus $f \in \hat{L}_\infty(0, c)$ implies that f is of bounded variation; the converse is obviously true. In this case (57) is replaced by

$$\int_{1/\alpha^\beta}^c \psi(t)\, h_0(\alpha, t)\, \mathrm{d}t = O(\alpha^{2P-\beta})$$

and (58) becomes
$$\int_0^{1/\alpha^\beta} \psi(t)\, h_0(\alpha, t)\, \mathrm{d}t = O(\alpha^{2P-\beta}).$$

Hence we obtain

THEOREM 16. *If* (50), (51) *and* (54) *hold, and if* f *is of bounded variation, then*

$$\int_0^c f(t)\, h_0(\alpha, t)\, \mathrm{d}t = O(\alpha^{2P-\beta})$$

as $\alpha \to +\infty$.

9. PRIMITIVES

In order to apply the theorems of the preceding section to generalized transforms it is necessary to ensure that h has similar properties to h_0, i.e. (50), (51) and either (52) or (54) must be verified. As far as (50) and (51) are concerned the conditions imposed on u_r are sufficient so that the main question is whether either (52) or (54) can be satisfied.

Let $h \in \mathscr{H}^+$ and suppose that, corresponding to a given $u_r(u_0 = h)$, there is a w_{r+1} such that for $x \geqslant 0$
$$\partial w_{r+1}/\partial x = u_r, \quad |w_{r+1}| = o\{\alpha^{Q_r}(1+x^2)^{S_1}\} \tag{59}$$
as $\alpha \to +\infty$, where
$$Q_r = 2M_0 + 2a_1 r - P_{0r}. \tag{60}$$
We might note that, if $h \in \mathscr{H}_1^+$,
$$Q_r = 2b_0 + (2c_1 - 1)\, r \tag{61}$$
and, if $h \in \mathscr{H}_2^+$,
$$Q_r = 2N - r. \tag{62}$$

Then $u_r \in \frac{1}{2}Q_r(0, \infty)$ and it will be said that u_r has a *primitive of the weak type*. In this case w_{r+1} complies with (52). If
$$|w_{r+1}| = O\{\alpha^{Q_{r+1}}(1+x^2)^{S_1}\} \tag{63}$$
as $\alpha \to +\infty$, (54) is satisfied and u_r is said to have a *primitive of the strong type*. Evidently, u_r has a primitive of the weak type when it possesses a primitive of the strong type, but the converse is not necessarily true.

It may happen that u_r has a primitive of the weak type not only for a particular value of r but also for all the values $r = R, R+1, \ldots$; when this occurs h will be described as *weakly primitive* and (R) will be added if it is necessary to indicate the first member of the sequence. Correspondingly, h will be said to be *strongly primitive* (R) if $u_R, u_{R+1}, \ldots$ all possess primitives of the strong type.

According to (59) $$\frac{\partial^{r+1} w_{r+1}}{\partial x^{r+1}} = \frac{\partial^r u_r}{\partial x^r} = h = \frac{\partial^{r+1} u_{r+1}}{\partial x^{r+1}}.$$

Therefore $$w_{r+1} - u_{r+1} = P_r \tag{64}$$

where P_r is a polynomial of degree r in x. If u_r has a primitive of the weak type (2) and (59) show that

$$|P_r| = o\{\alpha^{Q_r}(1+x^2)^M\}, \tag{65}$$

where M is the greater of S_1 and $M_{0,\,r+1}$, because

$$Q_{r+1} - Q_r = 2a_1 - P_{0,\,r+1} + P_{0,\,r} < 0$$

from (4) and $0 \leqslant a_1 < \frac{1}{2}$. Since P_r is a polynomial in x, only the coefficients involve α and therefore each coefficient must be $o(\alpha^{Q_r})$. Consequently

$$|\partial P_r/\partial x| = o\{\alpha^{Q_r}(1+x^2)^{M-\frac{1}{2}}\}. \tag{66}$$

Therefore (59) and a derivative of (64) give

$$|\partial u_{r+1}/\partial x - u_r| = o\{\alpha^{Q_r}(1+x^2)^{M-\frac{1}{2}}\}. \tag{67}$$

Conversely, if (67) holds so does (66) and then integration with respect to x gives a P_r satisfying (65). The w_{r+1} given by (64) then complies with (59) and u_r has a primitive of the weak type. Hence we have

Lemma 8. *u_r has a primitive of the weak type if and only if*

$$|\partial u_{r+1}/\partial x - u_r| = o\{\alpha^{Q_r}(1+x^2)^{M-\frac{1}{2}}\}$$

for some M, as $\alpha \to +\infty$.

In fact, since $\partial P_r/\partial x$ is a polynomial of degree $r-1$, M need not usually exceed $\frac{1}{2}r$.

A similar proof applies for a primitive of the strong type, the main difference being that $o(\alpha^{Q_r})$ is replaced by $O(\alpha^{Q_{r+1}})$ throughout. Hence

Lemma 9. *u_r has a primitive of the strong type if and only if*

$$|\partial u_{r+1}/\partial x - u_r| = O\{\alpha^{Q_{r+1}}(1+x^2)^{M-\frac{1}{2}}\}$$

for some M, as $\alpha \to +\infty$.

An immediate deduction from (2) and lemma 9 is that h always has a primitive of the strong type, namely u_1.

It has been seen in theorem 1 that $\chi h \in \mathscr{H}^+$ when $h \in \mathscr{H}^+$ and that U_r, as given by (6), plays the same role for χh that u_r plays for h. Now, from (7)

$$\frac{\partial U_{r+1}}{\partial x} - U_r = u_{r+1}(\alpha, 0)\, P_{r-1}(x) + \chi\left(\frac{\partial u_{r+1}}{\partial x} - u_r\right) + \frac{(-)^r}{(r-1)!}\int_0^x \left(\frac{\partial u_{r+1}}{\partial t} - u_r\right)\frac{\partial^r}{\partial t^r}\{(x-t)^{r-1}\chi(t)\}\,\mathrm{d}t.$$

The functions P_{r-1} and χ do not involve α and $u_{r+1}(\alpha, 0) = O(\alpha^{Q_{r+1}})$. It therefore follows, from lemma 8 (9), that U_r has a primitive of the weak (strong) type whenever u_r has. Thus multiplying h by χ does not affect the possession of a primitive, though it may alter the value of S_1 which is operative. One conclusion is

Theorem 17. *If $h \in \mathscr{H}^+$ and is weakly (strongly) primitive, then χh, where χ is defined in theorem 1, is also weakly (strongly) primitive (with the same R).*

As far as the particular spaces $\mathscr{H}_1^+$ and $\mathscr{H}_2^+$ are concerned, (10) and (11) imply that

$$\frac{\partial u_{r+1}}{\partial x} - u_r = \frac{r!}{(\mathrm{i}b\alpha)^{r+1}} \sum_{m=0}^{r-1} \frac{(-)^{m+1} x^{r-m-1}}{m!\,(r-m)!\,(r-l-m)!} \frac{\partial^{r-m}}{\partial x^{r-m}} j(\alpha, 0)$$
$$= O\{\alpha^{(2c_1-1)r+2b_0-1}(1+x^2)^{\frac{1}{2}(r-1)}\}$$

from (9). In this case (12) shows that

$$Q_r = 2M_0' + 2c_1 r - r = 2b_0 + (2c_1 - 1)\,r.$$

Since $0 > 2c_1 - 1 \geqslant -1$, the condition of lemma 9 is satisfied for $r = 0, 1, \ldots$. Thus h is strongly primitive when $h \in \mathscr{H}_1^+$.

If $h \in \mathscr{H}_2^+$, (23) and lemma 9 show that h is strongly primitive.

The combination of these results with theorem 17 gives

THEOREM 18. *If $h \in \mathscr{H}_1^+$, $\chi(x)\,h(\alpha, x)$ is strongly primitive (with $R = 0$). If $h \in \mathscr{H}_2^+$, $\chi(x)\,h(\alpha x)$ is strongly primitive (with $R = 0$).*

10. Separation of the critical points

In the discussion of the asymptotic behaviour of generalized transforms the class of generalized functions to be considered will be limited to those which are likely to occur in practical applications. Accordingly, the following assumption is made.

A. *The points where $g(\in K_+)$ is not infinitely differentiable (in the ordinary sense) are finite in number. These points, together with $+\infty$, are called critical points.*

In general, the critical points will include the origin, since the derivative in A is not intended to be one-sided. Let M be the number of critical points not at infinity and let A_j be an interval of length 2δ which has the jth critical point as mid-point. Let A_{M+1} be the interval which extends from $1/\delta$ to $+\infty$. Then δ can be chosen small enough for the A_j $(j = 1, \ldots, M+1)$ to be disjoint and to contain only one critical point each.

Let ϕ_j $(j = 1, \ldots, M)$ be a fine function which vanishes identically outside A_j and which equals 1 on the interval A_j' of length $2\delta_1$ $(< 2\delta)$ with the jth critical point as mid-point. On $A_j - A_j'$ let $0 \leqslant \phi_j(x) \leqslant 1$. Let $\eta_{M+1}(x)$ be an infinitely differentiable function such that $0 \leqslant \eta_{M+1} \leqslant 1$ and

$$\eta_{M+1}(x) = 1 \quad (x \geqslant 1/\delta_1),$$
$$= 0 \quad (x \leqslant 1/\delta).$$

The interval from $1/\delta_1$ to $+\infty$ is denoted by A'_{M+1}.

Consider the function η defined by

$$\eta(x) = \phi_1(x) + \phi_2(x) + \ldots + \phi_M(x) + \eta_{M+1}(x). \tag{68}$$

It is infinitely differentiable, takes the value 1 on $A_1' \cup A_2' \cup \ldots \cup A'_{M+1}$ and vanishes outside $A_1 \cup A_2 \cup \ldots \cup A_{M+1}$. Also $0 \leqslant \eta \leqslant 1$. Consequently, the function which is zero for $x \leqslant 0$ and equals $1 - \eta(x)$ for $x \geqslant 0$ is a fine function which vanishes in a neighbourhood of each critical point. Therefore, if $h \in \mathscr{H}^+$,

$$\int_0^\infty g(x)\,h(\alpha, x)\,\mathrm{d}x = \int_0^\infty g(x)\,\eta(x)\,h(\alpha, x)\,\mathrm{d}x + \int_0^\infty g(x)\,\{1 - \eta(x)\}\,h(\alpha, x)\,\mathrm{d}x.$$

Now g is infinitely differentiable away from the critical points and so $g(1-\eta)$ is a fine function which vanishes identically for $x \leqslant 0$. Therefore it can be regarded as a γ^+ and, by lemma 4,

$$\int_0^\infty g(x)\{1-\eta(x)\}h(\alpha,x)\,\mathrm{d}x$$

is good on $\alpha \geqslant 0$. Hence $$\lim_{\alpha\to+\infty} \alpha^r \int_0^\infty g(x)\{1-\eta(x)\}h(\alpha,x)\,\mathrm{d}x = 0$$

for any finite r. Thus we have

THEOREM 19. *If $g \in K_+$ and satisfies A, and if $h \in \mathscr{H}^+$ then*

$$\int_0^\infty g(x)\,h(\alpha,x)\,\mathrm{d}x = \int_0^\infty g(x)\,\eta(x)\,h(\alpha,x)\,\mathrm{d}x + O(\alpha^{-r})$$

for any finite r as $\alpha \to +\infty$.

Because of (68) this theorem indicates that, in many cases, the dominant asymptotic behaviour is the sum of contributions from neighbourhoods of the critical points.

The contribution from the critical point $+\infty$ is different in character from that due to the critical points at a finite distance. Therefore the critical point $+\infty$ must be handled separately from the other critical points; its contribution will be examined in the next section.

11. The contribution from infinity

The main object of the present section is to demonstrate that, under certain conditions, an order estimate of the contribution from infinity can be derived and that, in many circumstances, the contribution is negligible compared with that from other critical points.

Let χ_0 be a function of x which satisfies the conditions imposed on χ in theorem 1 and which is such that $1/\chi_0$ satisfies the same conditions, although the P_r need not be the same for both χ_0 and $1/\chi_0$. Typical χ_0 are 1, $\mathrm{e}^{\mathrm{i}ax}$ (a real) and $1+x^2$.

A generalized function g is said to *behave-S at positive infinity* if g is infinitely differentiable (in the ordinary sense) for $x \geqslant 1/\delta$ and $(1+x^2)^S\,[g(x)/\chi_0(x)]^{(l)}$ is absolutely integrable over $(1/\delta,\infty)$ for some χ_0 (with the properties of the preceding paragraph) and for some non-negative integer l (compare Jones (1966*a*)). If $(1+x^2)^S\,[g(x)/\chi_0(x)]^{(l+1)}$, $(1+x^2)^S\,[g(x)/\chi_0(x)]^{(l+2)}, \ldots$ are also absolutely integrable over $(1/\delta,\infty)$, g will be said to be *well-behaved-S at positive infinity*.

For example, $\mathrm{e}^{\mathrm{i}bx}x^\beta \ln x$ is well-behaved-S for any positive S, on choosing $\chi_0(x) = \mathrm{e}^{-\mathrm{i}bx}$ and taking l greater than $2S+1+\mathscr{R}(\beta)$. Similarly $x^\beta\,\mathrm{e}^{\mathrm{i}x^2}$ is well-behaved-S for any positive S by selecting $\chi_0 = \mathrm{e}^{-\mathrm{i}x^2}$.

Now, on account of theorems 1 and 8,

$$\int_0^\infty g(x)\,\eta_{M+1}(x)\,h(\alpha,x)\,\mathrm{d}x = \int_0^\infty (g/\chi_0)\,\eta_{M+1}\,\chi_0\,h\,\mathrm{d}x$$
$$= (-1)^l \int_0^\infty (\eta_{M+1}g/\chi_0)^{(l)}\,U_l\,\mathrm{d}x$$

on $\alpha > 0$, where U_r is defined in (6) but with χ_0 in place of χ. The derivative $(\eta_{M+1}g/\chi_0)^{(l)}$ consists of $\eta_{M+1}(g/\chi_0)^{(l)}$ plus a finite number of fine functions. Also, if ϕ is any fine function, identically zero on $x \leqslant 0$,

$$\int_0^\infty \phi(x)\,U_l(\alpha,x)\,\mathrm{d}x = (-)^s \int_0^\infty \phi^{(s)}(x)\,U_{l+s}(\alpha,x)\,\mathrm{d}x$$

by an application of theorem 8. In fact, the integral is over a finite interval and it follows from (8) that the right-hand side is $O(\alpha^{2M_0+2a_1(l+s)-P_{0,l+s}})$ as $\alpha \to +\infty$. By choosing s sufficiently large we can make this arbitrarily small because of (4). Hence, as $\alpha \to +\infty$,

$$\int_0^\infty \phi(x)\, U_l(\alpha, x)\, dx = O(\alpha^{-r}) \tag{69}$$

for any $r \geqslant 0$. Consequently

$$\int_0^\infty g(x)\, \eta_{M+1}(x)\, h(\alpha, x)\, dx = (-)^l \int_0^\infty \eta_{M+1}(g/\chi_0)^{(l)}\, U_l\, dx + O(\alpha^{-r}) \tag{70}$$

for any $r \geqslant 0$.

If u_l has a primitive of the weak type so does U_l (theorem 17) and, from (8),

$$|U_l| < C\alpha^{Q_l}(1+x^2)^{\mu_l}$$

as $\alpha \to +\infty$. Here $\quad \mu_l = \sup\,(M_{0l}+P_0, M_{0l}+P_1+\tfrac{1}{2}, \ldots, M_{0l}+P_l+\tfrac{1}{2}l).$

If g behaves-μ_l at positive infinity, $(1+x^2)^{\mu_l}(g/\chi_0)^{(l)}$ is absolutely integrable over $(1/\delta, \infty)$ and theorem 14 will be applicable. Hence we have proved

LEMMA 10. *If $h \in \mathscr{H}^+$, if g behaves-μ_l at positive infinity and if u_l possesses a primitive of the weak type*

$$\int_0^\infty g(x)\, \eta_{M+1}(x)\, h(\alpha, x)\, dx = o(\alpha^{Q_l})$$

as $\alpha \to +\infty$.

Q_l is defined in (60) and it is clear that the larger l can be made the smaller will be the contribution from infinity. A judicious choice of χ_0, which is at our disposal to some extent, may be helpful in keeping μ_l as low as possible.

A very useful χ_0 is e^{-idx} where d is real. For this χ_0, $P_r = 0$ for all r and $\mu_l = M_{0l}+\tfrac{1}{2}l$. This result, true for any $h \in \mathscr{H}^+$, can be improved for certain h. For example, if $h \in \mathscr{H}_1^+$, (12) and (13) imply that

$$Q_l = 2b_0 + (2c_1-1)\, l,$$

$$\mu_l = \sup\,(a_0', a_1'+\tfrac{1}{2}, \ldots, a_l'+\tfrac{1}{2}l),$$

$$a_r' = \sup\,(a_0, a_1, \ldots, a_r)$$

and so $\quad \mu_l = \tfrac{1}{2}l + \sup\,(a_0, a_1, \ldots, a_l).$

Further reduction in μ_l can be achieved when χ_0 is e^{-idx} and $h \in \mathscr{H}_1^+$. Replace b in (10) by $b+d/\alpha$; α can be chosen large enough for this to be non-zero. Then $\partial^r u_r/\partial x^r = h(\alpha, x)\, e^{-idx}$ and the estimate in (12) is unaffected as $\alpha \to +\infty$ because $b+d/\alpha \to b$. Thus $\mu_l = \sup\,(a_0, a_1+\tfrac{1}{2}, \ldots, a_l+\tfrac{1}{2}l)$. This can be expressed as

LEMMA 11. *If $h \in \mathscr{H}_1^+$ and $\chi_0 = e^{-idx}$ (d real), then $\mu_l = M_{0l}'$ in lemma* 10.

In general, there is nothing to prevent μ_l increasing as l increases and this imposes more constraint on the behaviour of g at infinity the larger l. However, it is possible that, for a given h, an appropriate χ_0 can be found such that $\mu_l \leqslant N$ no matter how large l is; such an h can be said to be of *limited growth.*

For instance, if $h \in \mathscr{H}_2^+$, put $b+d/\alpha$ for b in (16), i.e. think of $h(x)$ as $k(x)\, e^{-i(b+d/\alpha)x}$. Then (22) and (23) continue to hold. Inequalities (21) and (24) are also unaltered except that C_q now depends on α. However, this dependence on α is of no consequence since it arises from the presence of $b+d/\alpha$, which tends to b as $\alpha \to \infty$. Hence we have shown

LEMMA 12. *If $h \in \mathscr{H}_2^+$ and $\chi_0 = e^{-idx}$ (d real), h is of limited growth.*

If g is well-behaved-N and h is of limited growth it is evident that l can be taken as large as we like in lemma 10. Accordingly, we have

THEOREM 20. *If g is well-behaved-N, if h is of limited growth, then for any $r \geqslant 0$*

$$\int_0^\infty g(x)\, \eta_{M+1}(x)\, h(\alpha, x)\, dx = o(\alpha^{-r})$$

as $\alpha \to +\infty$.

On account of theorem 18 and lemma 12, theorem 20 will certainly be applicable when $h \in \mathscr{H}_2^+$.

These results permit estimates of the contribution to the asymptotic behaviour from the critical point at infinity. From some points of view theorem 20 is the most important because it shows that the contribution from infinity is negligible compared with contributions from other critical points so long as these are not exponentially small. However, in appropriate circumstances lemma 10 may be sufficient. For simplicity of writing most subsequent theorems will be based on theorem 20, but an additional clause using lemma 10 could always be added as in the following theorem which comes from theorem 19 and (68).

THEOREM 21. *Let $h \in \mathscr{H}^+$, $g \in K_+$ and satisfy A. If g is well-behaved-N, if h is of limited growth, then, for any finite $r \geqslant 0$,*

$$\int_0^\infty g(x)\, h(\alpha, x)\, dx = \sum_{j=1}^{M} \int_0^\infty g(x)\, \phi_j(x)\, h(\alpha, x)\, dx + O(\alpha^{-r})$$

as $\alpha \to +\infty$. If only the conditions of lemma 10 are met, $O(\alpha^{-r})$ is to be replaced by $o(\alpha^{Q_l})$.

Whenever the first half of theorem 20 is relevant, the dominant contribution to the asymptotic behaviour of a generalized transform can be expected to come from those points where g is not infinitely differentiable in the ordinary sense.

Suppose, in fact, that

$$\left[g(x) - \sum_{k=1}^{K} g_{jk}(x)\right]^{(N_j)}$$

is absolutely integrable over the interval where ϕ_j is non-zero and that g_{jk} $(k = 1, \ldots, K)$ is infinitely differentiable away from $x = x_j$. Assume that u_{N_j} has a primitive of the weak type. Then, by theorem 8,

$$\int_0^\infty \left\{g(x) - \sum_{k=1}^{K} g_{jk}(x)\right\} \phi_j(x)\, h(\alpha, x)\, dx = (-)^{N_j} \int_0^\infty \left[\left\{g(x) - \sum_{k=1}^{K} g_{jk}(x)\right\} \phi_j(x)\right]^{(N_j)} u_{N_j}(\alpha, x)\, dx.$$

The derivatives of ϕ_j vanish on an interval surrounding any point where g and g_{jk} are not infinitely differentiable. Therefore any term from the derivative in the integrand which contains a derivative of ϕ_j is a fine function and is covered by the estimate in (69). The remaining term involves

$$\left[g(x) - \sum_{k=1}^{K} g_{jk}(x)\right]^{(N_j)} \phi_j(x),$$

which is absolutely integrable and vanishes outside a finite interval. The assumption on u_{N_j} ensures that lemma 7 is applicable and so the integral on the right is $o(\alpha^{Q_{N_j}})$. Hence, with the given assumptions,

$$\int_0^\infty g(x)\, \phi_j(x)\, h(\alpha, x)\, dx = \sum_{k=1}^{K} \int_0^\infty g_{jk}(x)\, \phi_j(x)\, h(\alpha, x)\, dx + o(\alpha^{Q_{N_j}}). \qquad (71)$$

This formula is useful in enabling g to be approximated by suitable g_{jk}. The power Q_{Nj} which occurs is, of course, a maximum dictated by the inequalities of definition 1. For particular h it may, and usually will, be possible to make considerable reduction below this general estimate.

It may be possible to simplify the right-hand side of (71) in a very helpful fashion when circumstances are favourable. The analysis differs according as the critical point is the origin ($j = 1$) or not ($j \neq 1$). Suppose g_{1k} and h satisfy the first condition of theorem 21. Then

$$\int_0^\infty g_{1k}(x)\, h(\alpha, x)\, \mathrm{d}x = \int_0^\infty g_{1k}(x)\, \phi_1(x)\, h(\alpha, x)\, \mathrm{d}x + O(\alpha^{-r}) \tag{72}$$

for any $r \geqslant 0$, because g_{1k} has no critical point other than the origin. Substituting (72) in (71) we obtain

$$\int_0^\infty g(x)\, \phi_1(x)\, h(\alpha, x)\, \mathrm{d}x = \sum_{k=1}^{K} \int_0^\infty g_{1k}(x)\, h(\alpha, x)\, \mathrm{d}x + o(\alpha^{Q_{N_1}}). \tag{73}$$

This formula has the advantage of expressing the contribution of the origin to the asymptotic development in terms of the generalized transforms of the g_{1k} without the intervention of ϕ_1. As long as g_{1k} has a reasonably simple transform a good estimate of the asymptotic behaviour can be derived.

When $j \neq 1$, $H(x)\, g_{jk}(x)$ is well-defined because g_{jk} is infinitely differentiable in a neighbourhood of the origin. Thus $H(x)\, g_{jk}(x) \in K_+$ but it may have a critical point at the origin as well as the one at $x = x_j$. Hence theorem 21 will now give

$$\int_0^\infty g_{jk}(x)\, h(\alpha, x)\, \mathrm{d}x = \int_0^\infty g_{jk}(x)\, \{\phi_1(x) + \phi_j(x)\}\, h(\alpha, x)\, \mathrm{d}x + O(\alpha^{-r}).$$

Consequently, when $j \neq 1$,

$$\int_0^\infty g(x)\, \phi_j(x)\, h(\alpha, x)\, \mathrm{d}x = \sum_{k=1}^{K} \int_0^\infty g_{jk}(x)\, \{1 - \phi_1(x)\}\, h(\alpha, x)\, \mathrm{d}x + o(\alpha^{Q_{N_j}}) \tag{74}$$

as $\alpha \to +\infty$. In essence this states that, in calculating the asymptotic behaviour, the contribution from the origin to the generalized transform should be ignored. There is no difficulty in estimating the contribution due to the origin because g_{jk} is infinitely differentiable there (see theorem 23 below).

The combination of (73), (74) and (71) leads to

THEOREM 22. *Let $h \in \mathscr{H}^+$, be weakly primitive and of limited growth. Let $g \in K_+$, satisfy assumption A and be well-behaved-N. If g_{jk} ($k = 1, \ldots, K_j$) is well-behaved-N and infinitely differentiable away from $x = x_j$ and if, in some interval including x_j,*

$$\left[g - \sum_{j=1}^{K_j} g_{jk}\right]^{(N_j)}$$

is absolutely integrable, then

$$\int_0^\infty g(x)\, h(\alpha, x)\, \mathrm{d}x = \sum_{k=1}^{K_1} \int_0^\infty g_{1k}(x)\, h(\alpha, x)\, \mathrm{d}x + \sum_{j=2}^{M} \sum_{k=1}^{K_j} \int_0^\infty g_{jk}(x)\, \{1 - \phi_1(x)\}\, h(\alpha, x)\, \mathrm{d}x + o(\alpha^{N_0}), \tag{75}$$

as $\alpha \to +\infty$, where $N_0 = \max\,(N_1, N_2, \ldots, N_M)$.

If $\partial h/\partial \alpha$ has the same properties as h in theorem 22 we can replace h throughout (75) by $\partial h/\partial \alpha$. On account of theorem 7 this is equivalent to saying that a derivative of the asymptotic development (75) can be taken without increasing the error. The relative error might, of course, be increased and, if this happened, it would probably pay to undertake direct consideration of the derivative.

Even when g and g_{jk} are not well-behaved the estimate in (75) may still be correct, provided that lemma 10 can be used with a sufficiently large value of l.

12. Discontinuities in derivatives

One particular expansion which is of interest can be derived from the foregoing analysis. Suppose that in a neighbourhood of the origin $g(x) = H(x)f(x)$ where f and its first K derivatives are continuous functions on $x \geqslant 0$ and $f^{(K+1)}$ is absolutely integrable. Then

$$\left[H(x)\left\{f(x) - \sum_{k=0}^{K} \frac{x^k}{k!} f^{(k)}(0)\right\}\right]^{(K+1)},$$

where $f^{(k)}(0) = \lim_{x\to+0} f^{(k)}(x)$, is absolutely integrable, being $Hf^{(K+1)}$, and so we may take $g_{1k} = (x^k/k!)f^{(k)}(0)$. The function x^k is well behaved at positive infinity because its $(k+1)$th derivative vanishes and indeed putting $l = k+1$, $\chi_0 = 1$ in (70) shows that (72) is true. Hence, from (73), we have

Theorem 23. *If $f^{(k)}$ $(k = 0, \ldots, K)$ are continuous functions and $f^{(K+1)}$ is absolutely integrable on $0 \leqslant x \leqslant \delta$, and if u_{K+1} has a primitive of the weak type then*

$$\int_0^\infty f(x)\, \phi_1(x)\, h(\alpha, x)\, \mathrm{d}x = \sum_{k=0}^{K} \frac{f^{(k)}(0)}{k!} \int_0^\infty x^k h(\alpha, x)\, \mathrm{d}x + o(\alpha^{Q_{K+1}})$$

as $\alpha \to +\infty$.

At a critical point which is not the origin the situation is somewhat different. If, in a neighbourhood of $x = x_j$, $g = f$ where f and its first K derivatives are continuous, (74) gives

$$\int_0^\infty f(x)\, \phi_j(x)\, h(\alpha, x)\, \mathrm{d}x = \sum_{k=0}^{K} \frac{f^{(k)}(x_j)}{k!} \int_0^\infty (x - x_j)^k \{1 - \phi_1(x)\}\, h(\alpha, x)\, \mathrm{d}x + o(\alpha^{Q_{K+1}}).$$

Since $1 - \phi_1$ is essentially of the same form as η_{M+1} the argument just used above shows that each of the integrals on the right is $O(\alpha^{-r})$ for any $r \geqslant 0$. Hence we have shown

Theorem 24. *If $f^{(k)}$ $(k = 0, \ldots, K)$ are continuous functions and $f^{(K+1)}$ is absolutely integrable on $-\delta \leqslant x - x_j \leqslant \delta (j \neq 1)$, and if u_{K+1} has a primitive of the weak type*

$$\int_0^\infty f(x)\, \phi_j(x)\, h(\alpha, x)\, \mathrm{d}x = o(\alpha^{Q_{K+1}})$$

as $\alpha \to +\infty$.

If $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, K$ theorem 23 gives the same kind of behaviour as theorem 24. Both theorems demonstrate that the more derivatives of g that are continuous across the critical point the less dominant is the contribution from the critical point.

There is one method of calculating the terms in the series of theorem 23 which may be helpful. It was given under different circumstances by Willis (1948). It is not difficult to see that, on $\alpha > 0$,

$$\begin{aligned}\operatorname*{Lim}_{\mu\to+0} \frac{\mathrm{d}^k}{\mathrm{d}\mu^k} (-)^k \int_0^\infty \mathrm{e}^{-\mu x} h(\alpha, x)\, \mathrm{d}x &= \operatorname*{Lim}_{\mu\to+0} \int_0^\infty x^k\, \mathrm{e}^{-\mu x} h(\alpha, x)\, \mathrm{d}x \\ &= \int_0^\infty x^k h(\alpha, x)\, \mathrm{d}x\end{aligned}$$

from corollary 11*b*. Thus the terms in theorem 23 can be calculated from the generalized limit on the left. In particular, if

$$\int_0^\infty e^{-\mu x} h(\alpha, x)\, dx = \sum_{k=0}^{K} h_k(\alpha)\, \mu^k + o(\mu^K)$$

as $\mu \to +0$, then
$$\int_0^\infty f(x)\, \phi_1(x)\, h(\alpha, x)\, dx = \sum_{k=0}^{K} (-)^k f^{(k)}(0)\, h_k(\alpha) + (\alpha^{Q_{K+1}}). \tag{76}$$

13. Contribution from the origin

This section is concerned with evaluating the asymptotic behaviour furnished by the critical point at the origin when g cannot be represented by the simple series expansion of the preceding section. The easiest result occurs when $g = (Hf)^{(m)}$ on $(-\delta, \delta)$ where $f \in L_1(0, \delta)$. Then, by theorems 8 and 1,

$$\int_0^\infty g(x)\, \phi_1(x)\, h(\alpha, x)\, dx = (-)^m \int_0^\infty f(x) \frac{\partial^m}{\partial x^m} \{\phi_1(x)\, h(\alpha, x)\}\, dx$$
$$= (-)^m \int_0^\infty f(x)\, \phi_1(x) \frac{\partial^m}{\partial x^m} h(\alpha, x)\, dx + O(\alpha^{-r})$$

for any $r \geqslant 0$, by (69) because any derivative of ϕ_1 vanishes identically on an interval which includes the only possible critical point of f. Therefore, so long as $a_{m,0} > a_{m-1,0}$, lemma 7 gives

$$\int_0^\infty (Hf)^{(m)}\, \phi_1(x)\, h(\alpha, x)\, dx = o(\alpha^{2a_{m,0}}) \tag{77}$$

as $\alpha \to +\infty$. Equally well, if $g^{(m)}$ is absolutely integrable theorem 8 enables us to say that

$$\int_0^\infty g(x)\, \phi_1(x)\, h(\alpha, x)\, dx = o(\alpha^{Q_m}),$$

when u_m has a primitive of the weak type.

The analysis becomes more complicated when one wishes to use a series expansion for part of the integrand but not for the whole. Suppose that $g = (Hf)^{(m_1)}$ where $m_1 \leqslant m$ and that $x^s (Hf)^{(s)} \in L_1(-\delta, \delta)$ for $s = m_1, m_1+1, \ldots, m$. Let $K(x)$ be continuous and have m continuous derivatives on $0 \leqslant x \leqslant \delta$. Then

$$\int_0^\infty g(x)\, K(x)\, \phi_1(x)\, h(\alpha, x)\, dx = \sum_{s=0}^{m_1-1} \frac{K^{(s)}(0)}{s!} \int_0^\infty x^s g(x)\, \phi_1(x)\, h(\alpha, x)\, dx$$
$$+ \int_0^\infty x^{m_1} g(x)\, \phi_1(x) \left[\frac{K(x) - \sum_{s=0}^{m_1-1} K^{(s)}(0)\, x^s/s!}{x^{m_1}} \right] h(\alpha, x)\, dx.$$

The quantity in [ ] is continuously differentiable $m - m_1$ times and $x^s [x^{m_1} g(x)]^{(s)} \in L_1(-\delta, \delta)$ for $s = 0, 1, \ldots, m - m_1$. Therefore the $(m - m_1)$th derivative of

$$x^{m_1} g(x)\, \phi_1(x) \left[\frac{K(x) - \sum_{s=0}^{m_1-1} K^{(s)}(0)\, x^s/s!}{x^{m_1}} - \sum_{s=m_1}^{m-1} \frac{K^{(s)}(0)}{s!} x^{s-m_1} \right]$$

is absolutely integrable over $(-\delta, \delta)$. Hence, as in the derivation of (71), this quantity provides an integral of $o(\alpha^{Q_{m-m_1}})$ so long as u_{m-m_1} has a primitive of the weak type. We have therefore demonstrated

THEOREM 25. *If K is continuous with m continuous derivatives on $[0, \delta]$, if $g = (Hf)^{(m_1)}$ $(0 \leqslant m_1 \leqslant m)$ where $x^s(Hf)^{(s)} \in L_1(-\delta, \delta)$ for $s = m_1, m_1+1, \ldots, m$ and if u_{m-m_1} has a primitive of the weak type then*

$$\int_0^\infty g(x)\, K(x)\, \phi_1(x)\, h(\alpha, x)\, \mathrm{d}x = \sum_{s=0}^{m-1} \frac{K^{(s)}(0)}{s!} \int_0^\infty x^s g(x)\, \phi_1(x)\, h(\alpha, x)\, \mathrm{d}x + o(\alpha^{Q_{m-m_1}})$$

as $\alpha \to +\infty$. If, in addition, $x^{m-1}g$ is well-behaved-N and h is of limited growth, ϕ_1 on the right-hand side may be replaced by unity, provided that g has no other critical point than the origin.

A corresponding theorem, with a better estimate of the error, is supplied by theorem 15 or theorem 16 when suitable conditions are satisfied. Recalling lemma 9 we have

THEOREM 26. *If $K^{(s)}$ $(s = 0, 1, \ldots, m-1)$ is continuous and $K^{(m)}$ is of bounded variation on $[0, \delta]$, if $g = (Hf)^{(m_1)}$ $(0 \leqslant m_1 \leqslant m)$ where $x^s(Hf)^{(s)} \in \hat{L}_p(0, \delta)$ for $s = m_1, m_1+1, \ldots, m$ and if u_{m-m_1} has a primitive of the strong type then*

$$\int_0^\infty g(x)\, K(x)\, \phi_1(x)\, h(\alpha, x)\, \mathrm{d}x = \sum_{s=0}^{m-1} \frac{K^{(s)}(0)}{s!} \int_0^\infty x^s g(x)\, \phi_1(x)\, h(\alpha, x)\, \mathrm{d}x + R_m$$

as $\alpha \to +\infty$, where $R_m = o(\alpha^{Q_{m-m_1}-\beta+\beta/p})$ $(1 \leqslant p < \infty)$ and $R_m = O(\alpha^{Q_{m-m_1}-\beta})$ $(p = \infty)$ with $\beta = Q_{m-m_1} - Q_{m-m_1+1}$. Unity replaces ϕ_1 on the right under the same conditions as in theorem 25.

Neither of theorems 25 and 26 ensures that the terms in the series are more significant than the error term. Indeed, it is impossible to frame theorems of such generality and retain this certainty. However, in some cases, one can have a reasonable degree of confidence not only that the terms in the series dominate the error but also that they are likely to be of a decreasing order of magnitude.

Suppose first that $h \in \mathscr{H}_2^+$. On account of theorem 18, lemmas 7, 2 and 3

$$\int_0^\infty x^s (Hf)^{(s)}\, \phi_1(x)\, h^{(n)}(\alpha x)\, \mathrm{d}x = o(\alpha^{2N})$$

for any integer n, when $x^s(Hf)^{(s)} \in L_1(0, \delta)$. It is understood that, in this notation, $h^{(-r)}$ is the same as h_{-r} of § 2. Hence, from theorem 7,

$$\frac{\mathrm{d}^s}{\mathrm{d}\alpha^s} \int_0^\infty (Hf)^{(s)}\, \phi_1(x)\, h^{(n-s)}(\alpha x)\, \mathrm{d}x = o(\alpha^{2N})$$

or, so long as $s \geqslant m_1$,

$$\frac{\mathrm{d}^s}{\mathrm{d}\alpha^s} \alpha^{s-m_1} \int_0^\infty (Hf)^{(m_1)}\, \phi_1(x)\, h^{(n-m_1)}(\alpha x)\, \mathrm{d}x = o(\alpha^{2N})$$

from theorem 8 and the fact that any derivatives of ϕ_1 lead to the transforms of fine functions. Hence

$$\frac{\mathrm{d}^r}{\mathrm{d}\alpha^r} \alpha^{s-m_1} \int_0^\infty (Hf)^{(m_1)}\, \phi_1(x)\, h^{(n-m_1)}(\alpha x)\, \mathrm{d}x = o(\alpha^{2N+s-r})$$

for $0 \leqslant r \leqslant s$. It can be deduced at once that

$$\int_0^\infty x^p (Hf)^{(m_1)}\, \phi_1(x)\, h^{(n-m_1+p)}(\alpha x)\, \mathrm{d}x = o(\alpha^{2N+m_1-p}) \tag{78}$$

for $0 \leqslant p \leqslant s$. On choosing, for any given p, $n = m_1 - p$ we see that

$$\int_0^\infty x^s g(x)\, \phi_1(x)\, h(\alpha x)\, \mathrm{d}x = o(\alpha^{2N+m_1-s}) \tag{79}$$

for $0 \leqslant s \leqslant m$, when g satisfies the conditions of theorem 25 and $h \in \mathscr{H}_2^+$. The estimate (79) shows that, when $h \in \mathscr{H}_2^+$, the successive terms in the series in theorem 25 may be expected to be of decreasing order of magnitude and that the last term retained should be more significant than the error since, by (62), $Q_{m-m_1} = 2N + m_1 - m$ when $h \in \mathscr{H}_2^+$.

More generally, if u_{p-m_1} $(m_1 \leqslant p \leqslant m)$ has a primitive of the weak type,

$$\int_0^\infty [x^p g \phi_1]^{(p-m_1)}\, u_{p-m_1}\, \mathrm{d}x = o(\alpha^{Q_{p-m_1}})$$

because, when the derivative inside the integral is evaluated by Leibnitz's theorem, each term is absolutely integrable. It follows from theorem 8 that

$$\int_0^\infty x^p g(x)\, \phi_1(x)\, h(\alpha, x)\, \mathrm{d}x = o(\alpha^{Q_{p-m_1}})$$

for $m_1 \leqslant p \leqslant m$. Since Q_{p-m} decreases as p increases this provides the required result. On account of theorem 18 this result is true in particular when $h \in \mathscr{H}_1^+$ or $h \in \mathscr{H}_2^+$; some idea of the rapidity of decrease is given by (61) and (62).

14. Critical point not at the origin

When the critical point is not at the origin one method of dealing with it is to use a formula such as (74) and then attempt to employ the theory of the preceding section. There is, however, another way of proceeding which can be valuable. Since g is a generalized function there is a continuous f such that $g = f^{(r)}$ for some finite r. The functions $H(x-x_j)f(x)$ and $H(x_j-x)f(x)$ are well defined. Therefore

$$g = g_1 + g_2,$$

where

$$g_1(x) = \{H(x-x_j)f(x)\}^{(r)},$$

$$g_2(x) = \{H(x_j-x)f(x)\}^{(r)}.$$

Thus $g_1 = 0$ for $x < x_j$ and $g_2 = 0$ for $x > x_j$. The separation of g in this way is not unique because g_1 and g_2 could be replaced, for example, by $g_1 + \delta^{(p)}(x-x_j)$ and $g_2 - \delta^{(p)}(x-x_j)$ respectively, since this replacement leaves $g_1 + g_2$ unaltered and retains the properties $g_1 = 0$ for $x < x_j$, $g_2 = 0$ for $x > x_j$.

By theorem 10, with $\sigma = 1$ and $\tau = -x_j$,

$$\int_{x_j}^\infty g_1(x)\, \phi_j(x)\, h(\alpha, x)\, \mathrm{d}x = \int_0^\infty g_1(x+x_j)\, \phi_1(x)\, h(\alpha, x+x_j)\, \mathrm{d}x$$

and the critical point has now been converted to the origin so that all the theorems of the preceding section are available.

The same procedure can be adopted for the integral involving g_2, written as one from $-\infty$ to x_j, after changing the sign of x. Should it be desired to replace ϕ_1 eventually by unity it will be necessary to provide $h(\alpha, x)$ with an extension to negative values of x which has properties on $(-\infty, x_j)$ similar to those of h on (x_j, ∞).

In this way the discussion of the contribution from a critical point which is not at the origin can be subsumed under that for the critical point at the origin. The possibility that the behaviour of h at two critical points may be substantially different is examined in the next section.

15. Approximation to the kernel function

In some cases it may be desirable to make approximations to h in order to simplify the calculations. Therefore the effect of a straightforward expansion will now be considered.

Suppose that $h \in \mathscr{H}_2^+$; then the possibility exists that a reasonable approximation to the first few terms of the asymptotic development might be obtained by using the first terms of the Taylor expansion of k instead of k itself. If this is done in a neighbourhood of $x = x_j$ we obtain

$$\sum_{s=0}^{S-1} \frac{\alpha^s k^{(s)}(\alpha x_j)}{s!} \int_0^\infty (x-x_j)^s g(x)\,\phi_j(x)\,\mathrm{e}^{-\mathrm{i}b\alpha x}\,\mathrm{d}x. \tag{80}$$

Now, if $(x-x_j)^S g(x) \in L_1(x_j-\delta, x_j+\delta)$ we know, either from Jones (1966*a*) or from (79) (with $N=0$, $m_1 = S$), that the integral is $o(\alpha^{S-s})$ as $\alpha \to +\infty$. Hence, a typical term in the series is $o\{k^{(s)}(\alpha x_j)\,\alpha^S\}$ which is $o(\alpha^S)$ when $x_j = 0$ and $o(\alpha^{S+2N-2\delta s})$ when $x_j \neq 0$ from (14). Thus the terms in the series will probably be of decreasing order of magnitude when $x_j \neq 0$ and we may hope that a reasonable approximation can be derived in this way. When $x_j = 0$, however, all the terms seem to be of the same order of magnitude and it could not be anticipated that the expansion would be valuable.

Let us now attempt to assess the error introduced by using the approximation (80). Let k_* be defined by

$$k_* = k(\alpha x) - \sum_{s=0}^{S-1} \frac{\alpha^s}{s!}(x-x_j)^s k^{(s)}(\alpha x_j).$$

Then, an integration by parts gives

$$\int_{x_j}^{x} \frac{k_*\,\mathrm{e}^{-\mathrm{i}b\alpha x}}{\alpha^S(x-x_j)^S}\,\mathrm{d}x = \left[\frac{k_*\,\mathrm{e}^{-\mathrm{i}b\alpha x}}{-\mathrm{i}b\alpha^{S+1}(x-x_j)^S}\right]_{x_j}^{x} + \frac{1}{\mathrm{i}b}\int_{x_j}^{x}\left\{(x-x_j)\frac{\partial k_*}{\partial x} - Sk_*\right\}\frac{\mathrm{e}^{-\mathrm{i}b\alpha x}}{\alpha^{S+1}(x-x_j)^{S+1}}\,\mathrm{d}x.$$

By Taylor's theorem

$$\begin{aligned}
k_* &= \frac{\alpha^S}{S!}(x-x_j)^S k^{(S)}[\alpha\{x_j+\theta(x-x_j)\}] \\
&= \frac{\alpha^S}{S!}(x-x_j)^S k^{(S)}(\alpha x_j) + \frac{\alpha^{S+1}}{(S+1)!}(x-x_j)^{S+1} k^{(S+1)}[\alpha\{x_j+\theta_1(x-x_j)\}], \\
\frac{\partial k_*}{\partial x} &= \frac{\alpha^S(x-x_j)^{S-1}}{(S-1)!} k^{(S)}(\alpha x_j) + \frac{\alpha^{S+1}}{S!}(x-x_j)^S k^{(S+1)}[\alpha\{x_j+\theta_2(x-x_j)\}],
\end{aligned}$$

where $0 < \theta < 1$, $0 < \theta_{1,2} < 1$. Hence

$$\begin{aligned}
\int_{x_j}^{x} \frac{k_*\,\mathrm{e}^{-\mathrm{i}b\alpha x}}{\alpha^S(x-x_j)^S}\,\mathrm{d}x &= -\frac{\mathrm{e}^{-\mathrm{i}b\alpha x}}{S!\,\mathrm{i}b\alpha} k^{(S)}[\alpha\{x_j+\theta(x-x_j)\}] + \frac{\mathrm{e}^{-\mathrm{i}b\alpha x_j}}{\mathrm{i}b\alpha} k^{(S)}(\alpha x_j) \\
&\quad + \frac{1}{S!\,\mathrm{i}b}\int_{x_j}^{x}\left\{k^{(S+1)}[\alpha\{x_j+\theta_2(x-x_j)\}] - \frac{1}{S+1}k^{(S+1)}[\alpha\{x_j+\theta_1(x-x_j)\}]\right\}\mathrm{e}^{-\mathrm{i}b\alpha x}\,\mathrm{d}x \\
&= O(\alpha^{2N-2\delta S-1}) + O(\alpha^{2N-2\delta S-2\delta})
\end{aligned}$$

from (14), when $x_j \neq 0$ and x lies in the neighbourhood of x_j. On identifying $k_*/\alpha^S(x-x_j)^S$ and the integral with h_0 and h_{-1} respectively we see that (50) to (52) are satisfied. Since $(x-x_j)^S g \in L_1$, lemma 7 shows that the error in using the approximation (80) is $o(\alpha^{S+2N-2\delta S})$. Consequently, we can state

THEOREM 27. *If* $(x-x_j)^S g(x) \in L_1(x_j-\delta, x_j+\delta)$, $x_j \neq 0$ *and if* $h \in \mathscr{H}_2^+$ *then, when*

$$h(\alpha x) = k(\alpha x)\, e^{-ib\alpha x},$$

$$\int_0^\infty g(x)\,\phi_j(x)\,h(\alpha x)\,dx = \sum_{s=0}^{S-1} \frac{\alpha^S}{s!} k^{(S)}(\alpha x_j) \int_{-\infty}^\infty (x-x_j)^s g(x)\,\phi_j(x)\,e^{-ib\alpha x}\,dx + o(\alpha^{S+2N-2\delta S})$$

as $\alpha \to +\infty$.

It may be possible to improve the error estimate by taking advantage of theorem 15.

This theorem gives an approximation for the contribution from a critical point, which is not the origin, in terms of Fourier transforms. The Fourier transforms themselves can, of course, be estimated by the methods of § 13 (see also Jones 1966*a*).

Exactly the same technique may be employed when $h \in \mathscr{H}_1^+$ with the difference that the origin need not be excluded. Thus we have

THEOREM 28. *If* $(x-x_k)^S g(x) \in L_1(x_k-\delta, x_k+\delta)$ $(L_1(0,\delta)$ *if* $x_k = 0)$ *and if* $h \in \mathscr{H}_1^+$ *then, when* $h(\alpha, x) = j(\alpha, x)\, e^{-ib\alpha x}$,

$$\int_0^\infty g(x)\,\phi_k(x)\,h(\alpha, x)\,dx = \sum_{s=0}^{S-1} \frac{1}{s!} \frac{\partial^s j(\alpha, x_k)}{\partial x_k^s} \int_0^\infty (x-x_k)^s g(x)\,\phi_k(x)\,e^{-ib\alpha x}\,dx + o(\alpha^{2c_1 S+2b_0})$$

as $\alpha \to +\infty$.

A typical term in the series on the right-hand side is $o(\alpha^{S+(2c_1-1)s+2b_0})$ on account of (9) and the fact that the integral is $o(\alpha^{S-s})$. Thus the successive terms in the series may be expected to be of decreasing order of importance, but larger than the error term.

It may happen that g is sufficiently non-singular at a critical point for us to say that $(x-x_j)^S g^{(q)}(x) \in L_1(x_j-\delta, x_j+\delta)$. In that case the form of theorem 27 is not really suitable because it overestimates the size of the error term. However, we can use theorem 8 to write

$$\int_0^\infty g(x)\,h(\alpha x)\,dx = \frac{(-)^q}{\alpha^q} \int_0^\infty g^{(q)}(x)\,h_{-q}(\alpha x)\,dx.$$

It follows from lemma 3 and theorem 27 that

$$\int_0^\infty g(x)\,\phi_j(x)\,h(\alpha x)\,dx$$
$$= (-)^q \sum_{s=0}^{S-1} \frac{\alpha^{s-q}}{s!} k_{-q}^{(s)}(\alpha x_j) \int_{-\infty}^\infty (x-x_j)^S g^{(q)}(x)\,\phi_j(x)\,e^{-ib\alpha x}\,dx + o(\alpha^{S+2N-2\delta S-q}) \quad (81)$$

as $\alpha \to +\infty$. This form can be recast into one which is closer to that in theorem 27. Now

$$\frac{d}{dx} \sum_{s=0}^{S-1} \frac{\alpha^{s-q}}{s!} k_{-q}^{(s)}(\alpha x_j)\,(x-x_j)^s\,e^{-ib\alpha x} = -\frac{\alpha^{S-q}}{(S-1)!} k_{-q}^{(S)}(\alpha x_j)\,(x-x_j)^{S-1}\,e^{-ib\alpha x} + \sum_{s=0}^{S-1} \frac{\alpha^{s-q+1}}{s!} k_{-q+1}^{(s)}(\alpha x_j)\,(x-x_j)^s\,e^{-ib\alpha x}$$

on using (19). But, since $(x-x_j)^{S-1} g^{(q-1)} \in L_1$,

$$\alpha^{S-q} k_{-q}^{(S)}(\alpha x_j) \int_0^\infty (x-x_j)^{S-1} g^{(q-1)}(x)\,\phi_j(x)\,e^{-ib\alpha x}\,dx = o(\alpha^{S+2N-2\delta S-q}) \quad (82)$$

which is the same as the error term in (81). Therefore we can employ theorem 8 to reduce q to $q-1$ in (81) without affecting the error. Next observe that, if q is replaced by $q-1$ on the left-hand side of (82), we can use equation (26) of Jones (1966*a*) to demonstrate that the right-hand side of (82) is unaltered. Therefore we can repeat the process to reduce $q-1$ to $q-2$ in (81). Applying this result q times we obtain

THEOREM 29. *If* $(x-x_j)^{S-p}\, g^{(q-p)}(x) \in L_1(x_j-\delta, x_j+\delta)$, $x_j \neq 0$, *for* $p = 0, 1, \ldots, q$ *and if* $h \in \mathscr{H}_2^+$ *then, when* $h(\alpha x) = k(\alpha x)\,\mathrm{e}^{-\mathrm{i}b\alpha x}$, *the expansion of theorem 27 holds but the error term is replaced by* $o(\alpha^{S+2N-2\delta S-q})$.

It is not so easy to find such a satisfactory analogue of theorem 28. It is easily verified that

$$\left\{j(\alpha,x) - \sum_{s=0}^{S-1}\frac{1}{s!}\frac{\partial^s j(\alpha,x_k)}{\partial x_k^s}(x-x_k)^s\right\}\mathrm{e}^{-\mathrm{i}b\alpha x}/(x-x_k)^S$$

belongs to $\mathscr{H}_1^+$. The inequality (9) is satisfied with a_r, b_s replaced by a_{r+S} and b_s+c_1S respectively. The corresponding Q_r is $2b_0+2c_1S+(2c_1-1)\,r$. This means, as for (71), that

$$\int_0^\infty g\left\{j(\alpha,x) - \sum_{s=0}^{S-1}\frac{1}{s!}\frac{\partial^s j}{\partial x_k^s}(x-x_k)^s\right\}\mathrm{e}^{-\mathrm{i}b\alpha x}\,\mathrm{d}x = o\{\alpha^{2b_0+2c_1S+(2c_1-1)\,q}\}.$$

However the last term that is retained in the series is $O(\alpha^{2c_1S+2b_0-2c_1+1-q})$ so that the error estimate is not necessarily smaller than the estimate for the last term of the series. The reason is that each additional power of x in a Fourier transform reduces the asymptotic behaviour by $1/\alpha$, whereas the reduction for a generalized transform in $\mathscr{H}_1^+$ is $1/\alpha^{2c_1-1}$. For this reason we cannot expect, in general, to provide an error estimate which is smaller than the terms in the series.

16. EXAMPLES

It has been shown in §2 that $\mathrm{e}^{\mathrm{i}\alpha x^2} \in \mathscr{H}_+$. Also, for this h, $\partial u_r/\partial x = u_{r-1}$ so that, according to lemma 9, u_r has a primitive of the strong type for $r = 0, 1, \ldots$. Furthermore, $u_r = O\{(\alpha x)^{-r}\}$ for large x and so, when $\chi_0 = 1$, $\mu_r = -r$ with the consequence that h is certainly of limited growth. In addition $Q_r = -\tfrac{1}{2}r$.

It follows from theorem 20 that, if g is well-behaved-N (with $\chi_0 = 1$),

$$\int_0^\infty g(x)\,\eta_{M+1}(x)\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x = o(\alpha^{-r})$$

for any $r \geqslant 0$, as $\alpha \to +\infty$. Theorem 21 then shows that, if g has no critical point other than the origin, for any $r \geqslant 0$

$$\int_0^\infty g(x)\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x = \int_0^\infty g(x)\,\phi_1(x)\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x + O(\alpha^{-r}) \tag{83}$$

as $\alpha \to +\infty$.

Next, observe that, since $\mathrm{e}^{-\mu x^2}$ satisfies the conditions imposed on χ_μ in (44),

$$\int_0^\infty x^\lambda\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x = \operatorname*{Lim}_{\mu\to+0}\int_0^\infty x^\lambda\,\mathrm{e}^{\mathrm{i}\alpha x^2-\mu x^2}\,\mathrm{d}x$$

on $\alpha > 0$ when $\mathscr{R}(\lambda) > -1$. It is an immediate consequence that, when $\mathscr{R}(\lambda) > -1$,

$$\int_0^\infty x^\lambda\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x = \frac{(\tfrac{1}{2}\lambda-\tfrac{1}{2})!\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}(\lambda+1)}}{2\alpha^{\frac{1}{2}\lambda+\frac{1}{2}}} \tag{84}$$

on $\alpha > 0$. If $\mathscr{R}(\lambda) < -1$ but λ is not a negative integer we may use the fact that $x^\lambda H(x)$ is defined so that

$$\{x^\lambda H(x)\}' = \lambda x^{\lambda-1}H(x). \tag{85}$$

Then theorem 8 gives
$$\int_0^\infty x^{\lambda-1}\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x = -\frac{2\mathrm{i}\alpha}{\lambda}\int_0^\infty x^{\lambda+1}\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x$$
$$= (\tfrac{1}{2}\lambda-1)!\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}\lambda}/2\alpha^{\frac{1}{2}\lambda}$$

from (84). Evidently, therefore, (84) holds unless λ is a negative integer.

It may be shown in a similar way to that for (84) that, when $\mathscr{R}(\lambda) > -1$,

$$\int_0^\infty x^\lambda \ln x\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x = \frac{(\frac{1}{2}\lambda-\frac{1}{2})!\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}(\lambda+1)}}{4\alpha^{\frac{1}{2}\lambda+\frac{1}{2}}}\{\tfrac{1}{2}\pi\mathrm{i}+\psi(\tfrac{1}{2}\lambda-\tfrac{1}{2})-\ln\alpha\} \tag{86}$$

on $\alpha > 0$, where $\psi(x) = x!'/x!$. Now (85) holds even if λ is a negative integer (Jones 1966*b*) with the understanding that
$$x^{-1}H(x) = \{H(x)\ln x\}'+C\delta(x).$$

Hence, using (86) and theorem 8, we find that, on $\alpha > 0$,

$$\int_0^\infty x^{-2m-1}\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x = \frac{(\mathrm{i}\alpha)^m}{m!}(C'-\tfrac{1}{2}\ln\alpha) \tag{87}$$

for $m = 0, 1, 2, \ldots$. Here C' is arbitrary to the same extent that C is. On the other hand, if the integrand contains x^{-2m}, we obtain a formula which is the same as (84) with $\lambda = -2m$. Therefore we conclude that, on $\alpha > 0$,
$$\int_0^\infty x^\lambda\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x = \frac{(\frac{1}{2}\lambda-\frac{1}{2})!\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}(\lambda+1)}}{2\alpha^{\frac{1}{2}\lambda+\frac{1}{2}}} \tag{88}$$

when λ is not a negative odd integer. When λ is a negative odd integer (87) must be employed.

Corresponding results for integrands with logarithms can be deduced from (86) in a similar manner.

Suppose now that the function $K(x)$ is bounded by a polynomial, has no critical point other than the origin and that $K^{(s)}$ $(s = 0, 1, \ldots, m-1)$ is continuous while $K^{(m)}$ is of bounded variation on $[0, \delta]$. Then, if $g(x) = x^{\lambda_0-n}H(x)$ where $0 < \lambda_0 \leqslant 1$ and the positive integer n does not exceed $m+1$, the conditions of theorem 26 are met with $m_1 = n-1$ and $(1/p) > 1-\lambda_0$. Also $x^{m+\lambda_0-n}H(x)$ is well-behaved-N for some finite N and h is of limited growth. Thus the conditions given in theorem 25 for replacing ϕ_1 by unity are complied with. Consequently, if K has the properties just stated, (83), (88) and theorem 26 give, when $\lambda_0 \neq 1$,

$$\int_0^\infty x^{\lambda_0-n}K(x)\,\mathrm{e}^{\mathrm{i}\alpha x^2}\,\mathrm{d}x = \sum_{s=0}^{m-1}\frac{\{\frac{1}{2}(\lambda_0+s-n-1)\}!\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}(\lambda_0+s-n+1)}}{s!\,2\alpha^{\frac{1}{2}(\lambda_0+s-n+1)}}K^{(s)}(0)+o\{\alpha^{\frac{1}{2}(n-m-1-\lambda_0+\epsilon)}\} \tag{89}$$

as $\alpha \to +\infty$, where $\epsilon > 0$. If $\lambda_0 = 1$, the formula (87) must be used in those terms in which $1+s-n$ is an odd negative integer; the error term in this case is $O\{\alpha^{\frac{1}{2}(n-m)-1}\}$.

Similar expansions can be derived when $g(x) = x^{\lambda_0-n}H(x)\ln x$.

As an example of a kernel in $\mathscr{H}_1^+$ consider

$$\exp\{\mathrm{i}(\alpha+\nu)^\mu(x+1)^{\frac{1}{2}}-\mathrm{i}\alpha x\},$$

where $\nu > 0$, $0 < \mu < 1$. This satisfies (9) with $c_1 = \frac{1}{2}\mu$, $b_s = \frac{1}{2}s(\mu-1)$, $c_2 = \frac{1}{4}$, $a_r = -\frac{1}{4}r$. Now apply theorem 28; then, if $\mathscr{R}(\lambda) > -3$,

$$\int_0^\infty x^\lambda \exp\{\mathrm{i}(\alpha+\nu)^\mu(x+1)^{\frac{1}{2}}-\mathrm{i}\alpha x\}\,\mathrm{d}x = \exp\{\mathrm{i}(\alpha+\nu)^\mu\}\int_0^\infty\{x^\lambda+\tfrac{1}{2}(\alpha+\nu)^\mu x^{\lambda+1}\}\,\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}x+o(\alpha^{2\mu}).$$

It follows that, if $\lambda_0 \neq 1$,

$$\int_0^\infty x^{\lambda_0-3}\exp\{\mathrm{i}(\alpha+\nu)^\mu(x+1)^{\frac{1}{2}}-\mathrm{i}\alpha x\}\,\mathrm{d}x$$
$$= \exp\{\mathrm{i}(\alpha+\nu)^\mu-\tfrac{1}{2}\pi\mathrm{i}(\lambda_0-2)\}\{(\lambda_0-3)!\,\alpha^{2-\lambda_0}-(\lambda_0-2)!\,\tfrac{1}{2}\mathrm{i}\alpha^{1-\lambda_0}(\alpha+\nu)^\mu\}+o(\alpha^{2\mu}) \quad (90)$$

as $\alpha \to +\infty$. The error estimate could, of course, be improved by using theorem 15; it would then be $o(\alpha^{2\mu-\frac{1}{2}\lambda_0+\epsilon})$. It would also be possible to expand $(\alpha+\nu)^\mu$ asymptotically but it is sometimes found in practice that a formula will give better numerical results for the lower values of α if this is not done.

An illustration of a kernel in $\mathscr{H}_2^+$ is provided by the Bessel function $J_n(\alpha x)$ (n a non-negative integer), which obviously satisfies the conditions of definition 3 except possibly (14). The only difficulty with (14) arises when x is large and then, by expressing J_n in terms of the Hankel functions through

$$J_n(x) = \tfrac{1}{2}\{H_n^{(1)}(x)+H_n^{(2)}(x)\}$$

we see that (14) is satisfied with $\delta = \frac{1}{2}$ and $N = 0$ (in fact we could take $N = -\frac{1}{4}$ if it were not for the restriction in the definition). One could consider in a similar way $J_\nu(\alpha x)/(\alpha x)^\nu$.

Start with the known result (Watson 1944)

$$\int_0^\infty t^{\mu-1}J_n(\alpha t)\,\mathrm{e}^{-\epsilon^2t^2}\,\mathrm{d}t = \frac{(\frac{1}{2}n+\frac{1}{2}\mu-1)!\,(\frac{1}{2}\alpha)^n}{n!\,2\epsilon^{\mu+n}}\,\mathrm{e}^{-\alpha^2/4\epsilon^2}\,{}_1F_1(\tfrac{1}{2}n-\tfrac{1}{2}\mu+1;\,n+1;\,\alpha^2/4\epsilon^2) \quad (91)$$

when $\epsilon > 0$ and $\mathscr{R}(\mu) > -n$. Since

$${}_1F_1(a;c;x) = \frac{(c-1)!}{(a-1)!}\,\mathrm{e}^x x^{a-c}\{1+O(x^{-1})\}$$

as $x \to \infty$, corollary 11*b* gives

$$\int_0^\infty t^{\mu-1}J_n(\alpha t)\,\mathrm{d}t = \frac{(\frac{1}{2}\mu+\frac{1}{2}n-1)!\,2^{\mu-1}}{(\frac{1}{2}n-\mu)!\alpha^\mu} \quad (92)$$

subject to $\alpha > 0$, $\mathscr{R}(\mu) > -n$. The range of validity of this formula can be extended by employing (85) and theorem 8. Thus

$$\int_0^\infty t^{\mu-1}J_n(\alpha t)\,\mathrm{d}t = -\frac{1}{\mu}\int_0^\infty \alpha t^\mu J_n'(\alpha t)\,\mathrm{d}t.$$

The recurrence formula $zJ_n'(z) = nJ_n - zJ_{n+1}$ leads to

$$\int_0^\infty t^{\mu-1}J_n(\alpha t)\,\mathrm{d}t = \frac{\alpha}{\mu+n}\int_0^\infty t^\mu J_{n+1}(\alpha t)\,\mathrm{d}t$$

and, if (92) is employed to evaluate the right-hand side, (92) is recovered but with the restriction $\mathscr{R}(\mu) > -n-2$ so long as μ is not $-n$. Proceeding in this way we show that (92) is true for all complex μ except $-n, -n-2, \ldots$. Equation (92) is well known in the conventional theory of Bessel functions but is there subject to the restriction $-n < \mathscr{R}(\mu) < \frac{3}{2}$.

When $\mu = -n, -n-2, \ldots$ formulae analogous to (87) can be derived from

$$\int_0^\infty t^{\mu-1}J_n(\alpha t)\ln t\,\mathrm{d}t = \frac{(\frac{1}{2}\mu+\frac{1}{2}n-1)!\,2^{\mu-2}}{(\frac{1}{2}n-\frac{1}{2}\mu)!\,\alpha^\mu}\{\psi(\tfrac{1}{2}\mu+\tfrac{1}{2}n-1)+\psi(\tfrac{1}{2}n-\tfrac{1}{2}\mu)-2\ln\tfrac{1}{2}\alpha\},$$

which may be proved without difficulty.

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By lemma 12, $J_n(\alpha x)$ is of limited growth. It therefore follows from theorem 26 that, if K has the properties described in that theorem and has no critical point other than the origin (and is bounded by a polynomial at infinity),

$$\int_0^\infty x^{\lambda_0-q}K(x)\,J_n(\alpha x)\,\mathrm{d}x = \sum_{s=0}^{m-1}\frac{\{\tfrac{1}{2}(s+\lambda_0-q+n-1)\}!\,2^{s+\lambda_0-q}}{\{\tfrac{1}{2}(n-s-\lambda_0+q-1)\}!\,s!\,\alpha^{s+\lambda_0-q+1}}K^{(s)}(0)+o(\alpha^{q-1-m-\lambda_0+\epsilon}) \quad (93)$$

as $\alpha\to+\infty$, where $\epsilon>0$, $0<\lambda_0<1$ and q is a positive integer with $q\leqslant m+1$. If $\lambda_0=1$, (93) may have to be modified in a similar way to (89).

In contrast the application of theorem 27 gives, if $\lambda_0 \neq 1$,

$$\int_0^\infty (x-1)^{-\lambda_0-S}H(x-1)\,J_n(\alpha x)\,\mathrm{d}x$$
$$= \sum_{s=0}^{S-1}\frac{(s-\lambda_0-S)!}{s!\,\alpha^{1-\lambda_0-S}}\{h_1^{(s)}(\alpha)\,\mathrm{e}^{\mathrm{i}\alpha+\frac{1}{2}\pi\mathrm{i}(s-\lambda_0-S+1)}+h_2^{(s)}(\alpha)\,\mathrm{e}^{-\mathrm{i}\alpha-\frac{1}{2}\pi\mathrm{i}(s-\lambda_0-S+1)}\}+o(1) \quad (94)$$

as $\alpha\to+\infty$, where $\quad h_1(\alpha)=\tfrac{1}{2}H_0^{(1)}(\alpha)\,\mathrm{e}^{-\mathrm{i}\alpha},\quad h_2(\alpha)=\tfrac{1}{2}H_0^{(2)}(\alpha)\,\mathrm{e}^{\mathrm{i}\alpha}.$

By combining (93) and (94) one can obtain asymptotic formulae for integrals such as

$$\int_0^1 (1-x)^{-\frac{3}{2}}J_n(\alpha x)\,\mathrm{d}x.$$

In this connexion theorem 29 can sometimes be helpful.

Another function which is in $\mathscr{H}_2^+$ is $\mathrm{e}^{-\alpha x}$ (lemma 1). It is evident from theorem 28 that the dominant contribution to the asymptotic behaviour will come from the origin, any other critical point providing terms which are exponentially small. Expansions such as (71) should then enable the asymptotic behaviour to be expressed in terms of Laplace transforms of relatively simple generalized functions.

17. An example of inversion

As an illustration of the inversion theorem consider the Hankel transform. In order to verify (45) it is necessary to consider

$$\int_0^\infty y\alpha J_n(\alpha y)\int_0^\infty \gamma_m^+(x)\,J_n(\alpha x)\,\mathrm{d}x\,\mathrm{d}\alpha.$$

The inner integral is a good function of α and so the integral with respect to α converges absolutely. It may therefore be calculated as

$$\lim_{\epsilon\to+0}\int_0^\infty \mathrm{e}^{-\epsilon\alpha^2}y\alpha J_n(\alpha y)\int_0^\infty \gamma^+(x)\,J_n(\alpha x)\,\mathrm{d}x\,\mathrm{d}\alpha.$$

The order of integration can be interchanged in this integral and, since

$$\int_0^\infty \mathrm{e}^{-\epsilon\alpha^2}\alpha J_n(\alpha y)\,J_n(\alpha x)\,\mathrm{d}\alpha=\frac{1}{2\epsilon}I_n\left(\frac{xy}{2\epsilon}\right)\mathrm{e}^{-(x^2+y^2)/4\epsilon}$$

(Watson 1944), we obtain $\qquad \dfrac{y}{2\epsilon}\displaystyle\int_0^\infty \gamma^+(x)\,I_n\left(\frac{xy}{2\epsilon}\right)\mathrm{e}^{-(x^2+y^2)/4\epsilon}\,\mathrm{d}x.$

Also $\qquad \dfrac{y}{2\epsilon}\displaystyle\int_0^\infty I_n\left(\frac{xy}{2\epsilon}\right)\mathrm{e}^{-(x^2+y^2)/4\epsilon}\,\mathrm{d}x=\frac{\pi^{\frac{1}{2}}y}{2\epsilon^{\frac{1}{2}}}\mathrm{e}^{-y^2/8\epsilon}I_{\frac{1}{2}n}\left(\frac{y^2}{8\epsilon}\right)$

(Watson 1944). Hence the value of our integral is

$$\frac{y}{2\epsilon}\int_0^\infty \{\gamma^+(x)-\gamma^+(y)\}\, I_n\left(\frac{xy}{2\epsilon}\right) \mathrm{e}^{-(x^2+y^2)/4\epsilon}\,\mathrm{d}x + \frac{\pi^{\frac12}y}{2\epsilon^{\frac12}}\,\gamma^+(y)\,\mathrm{e}^{-y^2/8\epsilon}\, I_{\frac12 n}\left(\frac{y^2}{8\epsilon}\right).$$

If $y \neq 0$, the asymptotic formula for the modified Bessel function makes the last term tend to $\gamma^+(y)$ as $\epsilon \to 0$. If $y = 0$ the last term is zero (in fact, even if $y = \epsilon^{\frac12}$, the last term tends to zero as $\epsilon \to 0$ because of the presence of $\gamma^+(y)$). It is therefore legitimate to take the limit of the last term as $\gamma^+(y)$ when $\epsilon \to 0$. With regard to the term containing the integral it is zero when y is zero. If y is not zero split the interval of integration into the two portions $(0, (\epsilon/y)^{\frac12})$ and $((\epsilon/y)^{\frac12}, \infty)$. Now

$$\left|\frac{y}{2\epsilon}\int_{(\epsilon/y)^{\frac12}}^\infty \{\gamma^+(x)-\gamma^+(y)\}\, I_n\left(\frac{xy}{2\epsilon}\right) \mathrm{e}^{-(x^2+y^2)/4\epsilon}\,\mathrm{d}x\right|$$

$$< \frac{y}{2\epsilon}\,(\sup|\gamma^{+\prime}|)\int_{(\epsilon/y)^{\frac12}}^\infty |x-y| \left(\frac{2\epsilon}{y}\right)^{\frac12} \mathrm{e}^{-(x-y)^2/4\epsilon}\,\mathrm{d}x$$

$$< (\tfrac12 y\epsilon)^{\frac12}\,(\sup|\gamma^{+\prime}|)\int_{y^{-\frac12}-y/\epsilon^{\frac12}}^\infty |t|\,\mathrm{e}^{-\frac14 t^2}\,\mathrm{d}t$$

$$= O(\epsilon^{\frac12})$$

and

$$\left|\frac{y}{2\epsilon}\int_0^{(\epsilon/y)^{\frac12}} \{\gamma^+(x)-\gamma^+(y)\}\, I_n\left(\frac{xy}{2\epsilon}\right) \mathrm{e}^{-(x^2+y^2)/4\epsilon}\,\mathrm{d}x\right|$$

$$< \frac{y}{2\epsilon}\,(\sup|\gamma^{+\prime}|)\int_0^{(\epsilon/y)^{\frac12}} |x-y| \left(\frac{xy}{2\epsilon}\right)^n \mathrm{e}^{-(1/4\epsilon)\,(x-y)^2}\,\mathrm{d}x$$

$$< \left(\frac{y}{2\epsilon}\right)^{\frac12 n+\frac12} (\sup|\gamma^{+\prime}|)\,\mathrm{e}^{-(1/4\epsilon)\,\{(\epsilon/y)^{\frac12}-y\}^2}$$

$$\to 0$$

as $\epsilon \to 0$. Consequently, it has been shown that

$$\int_0^\infty y\alpha J_n(\alpha y)\int_0^\infty \gamma^+(x)\, J_n(\alpha x)\,\mathrm{d}x\,\mathrm{d}\alpha = \gamma^+(y)$$

on $y \geqslant 0$.

The theory developed above may now be applied. For example, since

$$\left[\frac{\partial^m}{\partial x^m} J_n(\alpha x)\right]_{x=0} = 0 \quad (m < n \quad \text{or} \quad m-n \text{ odd})$$

$$= \frac{m!\,(-)^{\frac12(m-n)}\,(\frac12\alpha)^m}{(\frac12 m-\frac12 n)!\,(\frac12 m+\frac12 n)!} \quad (m = n+2r)$$

(46) gives

$$y\int_0^\infty \alpha^{n+2r+1} J_n(\alpha y)\,\mathrm{d}\alpha = 0$$

on $y > 0$, for $r = 0, 1, \ldots$. This is consistent with (92).

There is also the general result

$$g(y) = \operatorname*{Lim}_{\mu\to+0}\int_0^\infty \alpha y J_n(\alpha y)\, H(\alpha)\int_0^\infty g(x)\,\mathrm{e}^{-\mu x} J_n(\alpha x)\,\mathrm{d}x\,\mathrm{d}\alpha$$

on $y > 0$, as a consequence of (49). The factor $\mathrm{e}^{-\mu x}$ can, of course, be replaced by a factor with similar properties, e.g. $\mathrm{e}^{-\mu x^2}$.

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The same technique that was used to prove the validity of (92) for all values of μ except $-n, -n-2, \dots$ may also be employed to establish (91) for the same values of μ. Accordingly, on $y > 0$,

$$y^{\mu-1} = \lim_{\epsilon\to+0}\int_0^\infty \frac{(\frac{1}{2}n+\frac{1}{2}\mu-1)!\,(\frac{1}{2}\alpha)^{n+1}}{n!\,\epsilon^{\mu+n}}\, y\, e^{(-\alpha^2/4\epsilon^2)} J_n(\alpha y)\, {}_1F_1\left(\tfrac{1}{2}n-\tfrac{1}{2}\mu+1;\, n+1;\, \frac{\alpha^2}{4\epsilon^2}\right) d\alpha$$

if $\mu \neq -n, -n-2, \dots$. This should be compared with a related result which can be deduced from (92), namely

$$y^{\mu-2} = \frac{(\frac{1}{2}n+\frac{1}{2}\mu-1)!\,2^{\mu-1}}{(\frac{1}{2}n-\frac{1}{2}\mu)!}\int_0^\infty x^{1-\mu} J_n(\alpha x)\, dx$$

if $\mu \neq n+2, n+4, \dots$.

Appendix A

This appendix is concerned with the properties of certain singular integrals when a weak function is present in the integrand. Some discussion of such integrals has been given elsewhere (Jones 1966*c*) but, for convenience, the main results that are needed will be summarized briefly below.

Only integrals over a finite interval will be considered. Since weak functions and generalized functions are the same on a finite interval the results derived are equally applicable to integrals of generalized functions.

Let ψ be a complex-valued function such that ψ and its first $(r-1)$ ordinary derivatives are continuous on the finite interval $[a, b]$. Let $\psi^{(r)} \in L_1(a, b)$, i.e. $\psi^{(r)}$ is absolutely integrable over (a, b).

There are extensions of ψ onto a larger interval, which have the same properties. For example, construct a polynomial P_{r-1} of degree $r-1$ on $x \leqslant a$ such that $P_r^{(s)} = \psi^{(s)}$ $(s = 0, 1, \dots, r-1)$ at $x = a$. Multiply P_{r-1} by a fine function which is unity for $a-\frac{1}{8}\epsilon \leqslant x \leqslant b+\frac{1}{8}\epsilon$ $(\epsilon > 0)$ and identically zero for $x \leqslant a-\frac{1}{4}\epsilon$. Then, if this product is called ψ on $x < a$ an extension has been provided which has the desired properties and also vanishes identically for $x \leqslant a-\frac{1}{4}\epsilon$. Clearly, a similar extension can be supplied on $x > b$.

The space of functions with the properties of ψ, extended so as to vanish outside $(a-\frac{1}{4}\epsilon, b+\frac{1}{4}\epsilon)$, will be denoted by $L^r(a, b)$.

Let w be a weak function which is zero on $x < a$ and on $x > b$. Such a weak function can be expressed, on $a-\frac{1}{2}\epsilon < x < b+\frac{1}{2}\epsilon$, as $w = f^{(r)}$ for some finite r, where f is continuous and vanishes outside $(a-\epsilon, b+\epsilon)$. The space of all weak functions which can be represented in this way for a given r will be denoted by $W^r(a, b)$. Obviously, if $w \in W^r(a, b)$ then $w \in W^{r+1}(a, b)$.

If $w \in W^r(a, b)$, then $f^{(r)} = 0$ on $a-\frac{1}{2}\epsilon < x < a$ so that f must be a polynomial of degree $r-1$ on this interval.

If $w \in W^r(a, b)$ and $\psi \in L^r(a, b)$ define

$$\int_a^b w(x)\,\psi(x)\,dx = (-)^r\int_{a-\epsilon}^{b+\epsilon} f(x)\,\psi^{(r)}(x)\,dx. \tag{A 1}$$

Since the integrand involves only ordinary functions, f being continuous and $\psi^{(r)}$ integrable, the integral on the right is one in the conventional sense.

It should be remarked that this definition is, in fact, independent of ϵ. For, on account of the properties of ψ,

$$\int_{a-\epsilon}^{a} f(x)\,\psi^{(r)}(x)\,dx = [f(x)\,\psi^{(r-1)}(x)]_{a-\frac{1}{4}\epsilon}^{a} - \int_{a-\frac{1}{4}\epsilon}^{a} f'(x)\,\psi^{(r-1)}(x)\,dx$$

by an integration by parts. The value of the first term is $f(a)\,\psi^{(r-1)}(a)$ because $\psi^{(r-1)}$ vanishes at the lower limit and f is a polynomial of degree $(r-1)$ on the interval under consideration. Repeating the process $r-1$ times we obtain

$$\int_{a-\epsilon}^{a} f(x)\,\psi^{(r)}(x)\,\mathrm{d}x = [f\psi^{(r-1)} - f'\psi^{(r-2)} + \dots + (-)^{r-1} f^{(r-1)}\psi]_{x=a-0}. \tag{A 2}$$

For $\psi, \psi', \dots, \psi^{(r-1)}$ it does not matter whether we put $x = a-0$ or $x = a$ since they are continuous across $x = a$; however, it can be significant for f because we can be sure of finite values for the derivatives only on $x < a$. Similar considerations apply to the interval $(b, b+\epsilon)$ and so (A 1) does not depend upon ϵ, nor the values of ψ outside $[a, b]$.

As an illustration, suppose that $w = \{(x-a)\,H(x-a)\,\phi(x)\}''$ where ϕ is a fine function which is unity for $a-\frac{1}{2}\epsilon < x < b+\frac{1}{2}\epsilon$ and zero for $x \leqslant a-\epsilon$, $x \geqslant b+\epsilon$. Then $w \in W^2(a, b)$ and, indeed, $w = \delta(x-a)$. After two integration by parts, (A 1) gives

$$\int_a^b w(x)\,\psi(x)\,\mathrm{d}x = \phi(a)\,\psi(a) + \int_a^{b+\epsilon} \{2\phi' + (x-a)\,\phi''\}\,\psi\,\mathrm{d}x$$
$$= \phi(a)\,\psi(a)$$

since ϕ', ϕ'' are zero for $a \leqslant x < b+\frac{1}{2}\epsilon$ and ψ is identically zero for $x \geqslant b+\frac{1}{4}\epsilon$. Since $\phi(a) = 1$ we have

$$\int_a^b \delta(x-a)\,\psi(x)\,\mathrm{d}x = \psi(a) \tag{A 3}$$

when $\psi \in L^2(a, b)$. More generally

$$\int_a^b \delta^{(m)}(x-a)\,\psi(x)\,\mathrm{d}x = (-)^m \psi^{(m)}(a) \tag{A 4}$$

when $\psi \in L^{m+2}(a, b)$. Actually the conditions on ψ can be lightened by observing that (A 1) could still be used if $f \in L_1(a, b)$ so long as $\psi^{(r)}$ is continuous. There would be no alteration to (A 2), but now (A 4) would be valid provided that $\psi^{(m+1)}$ were continuous. Still less restriction would be imposed if a Stieltjes integral were used to define the left-hand side of (A 3).

Assume that $w_m \in W^r(a, b)$, $w \in W^r(a, b)$ so that $w_m = f_m^{(r)}$, $w = f^{(r)}$. Suppose that $\lim_{m\to\infty} f_m = f$ uniformly. Then, if $\psi \in L^r(a, b)$,

$$\lim_{m\to\infty} \int_a^b w_m(x)\,\psi(x)\,dx = \lim_{m\to\infty} (-)^r \int_{a-\epsilon}^{b+\epsilon} f_m(x)\,\psi^{(r)}(x)\,\mathrm{d}x$$
$$= (-)^r \int_{a-\epsilon}^{b+\epsilon} f(x)\,\psi^{(r)}(x)\,\mathrm{d}x$$

since the convergence is uniform. Hence we have proved

THEOREM A 1. *If $w_m = f_m^{(r)} \in W^r(a, b)$, if $w = f^{(r)} \in W^r(a, b)$ and $\lim_{m\to\infty} f_m = f$ uniformly then*

$$\lim_{m\to\infty} \int_a^b w_m(x)\,\psi(x)\,\mathrm{d}x = \int_a^b w(x)\,\psi(x)\,\mathrm{d}x \tag{A 5}$$

when $\psi \in L^r(a, b)$.

On the other hand, if $\lim_{m\to\infty} w_m = w$ in the weak sense there is r_1 such that $w_m = f_m^{(r_1)}$, $w = f^{(r_1)}$ and

$f_m \to f$ uniformly. Hence, provided that $\psi \in L^{r_1}(a, b)$, (A 5) will be valid. Consequently, there has been demonstrated

Corollary A 1. *If w_m and w are zero for $x < a$ and $x > b$ (a and b finite), and if $\lim_{m\to\infty} w_m = w$ in the weak sense then*

$$\lim_{m\to\infty} \int_a^b w_m(x)\, \psi(x)\, \mathrm{d}x = \int_a^b w(x)\, \psi(x)\, \mathrm{d}x$$

provided that $\psi \in L^r(a, b)$ for sufficiently large r.

Appendix B

In this appendix we show that the sequences $\{\gamma_n^+\}$ define the space K_+, of generalized function which are zero for $x < 0$.

First, if $\{\gamma_n^+\}$ is a regular sequence, each γ_n^+ is identically zero for $x \leqslant 0$ and so defines a generalized function which is zero for $x < 0$, i.e. $g \in K_+$.

Conversely, if $g \in K_+$, $g = f^{(r)}$ where $f \in K_1$ (Jones 1966 *b*). Since $f^{(r)} = 0$ for $x < 0$, f must be a polynomial on $x < 0$. If this polynomial is subtracted from f the representation of g is unaltered but now f is zero for $x < 0$. Hence, when $g \in K_+$, we can write $g = f^{(r)}$ where $f \in K_1$ and is zero for $x < 0$.

Suppose now that $f \in K_1$ and is zero for $x < 1$. Then the sequence

$$\left\{\int_1^\infty f(t)\, \rho\{n(t-x)\}\, n\, \mathrm{e}^{-t^2/n^2}\, \mathrm{d}t\right\},$$

where

$$\rho(x) = \frac{\mathrm{e}^{-1/(1-x^2)}}{\int_{-1}^{1} \mathrm{e}^{-1/(1-t^2)}\, \mathrm{d}t} \quad (|x| < 1)$$
$$= 0 \quad (|x| \geqslant 1),$$

is a sequence of good functions which defines f (see Jones 1966 *b*) and moreover each of the good functions vanishes identically for $x \leqslant 0$ because of the properties of ρ. Hence such an f is certainly defined by a sequence of the type $\{\gamma_n^+\}$.

It remains to discuss the case of an integrable f which is zero for $x < 0$ and for $x > 1$. Consider

$$\phi_n(x) \int_0^1 f(t)\, \rho\{n(t-x)\}\, n\, \mathrm{d}t,$$

where

$$\phi_n(x) = \mathrm{e}^{-1/nx} \quad (x > 0)$$
$$= 0 \quad (x \leqslant 0).$$

Obviously this is a fine function of x which vanishes identically for $x \leqslant 0$ and so can be regarded as a γ_n^+. Also, if γ is any good function,

$$\int_{-\infty}^{\infty} \gamma(x)\, \phi_n(x) \int_0^1 f(t)\, \rho\{n(t-x)\}\, n\, \mathrm{d}t\, \mathrm{d}x = \int_{-\infty}^{\infty} \gamma(x) \int_0^1 f(t)\, \rho\{n(t-x)\}\, n\, \mathrm{d}t\, \mathrm{d}x$$
$$+ \int_{-\infty}^{\infty} \gamma(x)\, \{\phi_n(x) - 1\} \int_0^1 f(t)\, \rho\{n(t-x)\}\, n\, \mathrm{d}t\, \mathrm{d}x. \quad \text{(B 1)}$$

The first term on the right-hand side tends to

$$\int_0^1 f(x)\, \gamma(x)\, \mathrm{d}x$$

as $n\to\infty$. As regards the second term the change of variable $x = t-y/n$ leads to

$$\int_{-1}^{1}\rho(y)\int_{0}^{1}f(t)\,\gamma\left(t-\frac{y}{n}\right)\left\{\phi_n\left(t-\frac{y}{n}\right)-1\right\}\mathrm{d}t\,\mathrm{d}y.$$

Now since ϕ_n never exceeds unity

$$\left|\int_{0}^{1/n^{\frac{1}{2}}}f(t)\,\gamma\left(t-\frac{y}{n}\right)\left\{\phi_n\left(t-\frac{y}{n}\right)-1\right\}\mathrm{d}t\right| \leqslant 2\int_{0}^{1/n^{\frac{1}{2}}}\left|f(t)\,\gamma\left(t-\frac{y}{n}\right)\right|\mathrm{d}t$$

$$\leqslant 2B\int_{0}^{1/n^{\frac{1}{2}}}|f(t)|\,\mathrm{d}t,$$

where B is an upper bound for $\gamma(x)$ in, say, $(-2,2)$. As $n\to\infty$ the last term tends to zero by a well-known property of integrable functions.

Also, when $t\geqslant 1/n^{\frac{1}{2}}$ and $n>1$, $t>y/n$ since y does not exceed unity. Therefore

$$\left|\phi_n\left(t-\frac{y}{n}\right)-1\right| = \left|\exp\left(\frac{-1}{nt-y}\right)-1\right| < \frac{1}{nt-y} < \frac{1}{n^{\frac{1}{2}}-1}.$$

Hence $$\left|\int_{1/n^{\frac{1}{2}}}^{1}f(t)\,\gamma\left(t-\frac{y}{n}\right)\left\{\phi_n\left(t-\frac{y}{n}\right)-1\right\}\mathrm{d}t\right| < \frac{B}{n^{\frac{1}{2}}-1}\int_{1/n^{\frac{1}{2}}}^{1}|f(t)|\,\mathrm{d}t < \frac{B}{n^{\frac{1}{2}}}\int_{0}^{1}|f(t)|\,\mathrm{d}t\to 0$$

as $n\to\infty$.

Consequently, the second term on the right-hand side of (B 1) tends to zero as $n\to\infty$ and

$$\int_{-\infty}^{\infty}\gamma(x)\,\phi_n(x)\int_{0}^{1}f(t)\,\rho\{n(t-x)\}\,n\,\mathrm{d}t\,\mathrm{d}x\to\int_{0}^{1}f(x)\,\gamma(x)\,\mathrm{d}x.$$

Thus it has been shown that, when $f\in K_1$ and is zero for $x<0$, there is a regular sequence $\{\gamma_n^+\}$ defining f. Since the sequence $\{\gamma_n^{+(r)}\}$ is regular and defines $f^{(r)}$ we see that, because $\gamma_n^{+(r)}$ vanishes identically for $x\leqslant 0$, any $g\in K_+$ can be defined by a regular sequence of good functions which are zero for $x\leqslant 0$.

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